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A singular minimizer of a smooth strongly convex functional in three dimensions

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1 Introduction

We consider variational integrals of the form,

$$I(u) = \int_{\Omega} f(Du(x))dx, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set, $u\colon \Omega \to \mathbf{R}^m$ is a mapping belonging to $W^{1,2}(\Omega)$, $Du(x)$ denotes the gradient matrix of u at $x \in \Omega$, and f is a function defined on the set $M^{m\times n}$ of all real $m \times n$ matrices satisfying the following assumptions:

f *is smooth, strongly convex with uniformly bounded second derivatives*. $(*)$

We recall that f is said to be strongly convex if there exists a constant $\nu > 0$, such that for all $\xi \in M^{m\times n}$, $X \in M^{m\times n}$, the inequality $f_{p^i_\alpha p^j_\beta}(X)\xi^i_\alpha \xi^j_\beta \geq \nu|\xi|^2$ holds. Here and in what follows we will be using Einstein's summation convention.

We shall consider the regularity of minimizers of I in $W^{1,2}(\Omega)$. Here by a minimizer we mean a function $u \in W^{1,2}(\Omega)$ such that for any smooth function $\phi\colon \Omega \to \mathbf{R}^m$ compactly supported in Ω the inequality $I(u+\phi) \geq I(u)$ holds. When f satisfies $(*)$, it is not difficult to see that u is a minimizer

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of I if and only if u is a weak solution of the $Euler - Lagrange$ equation of I, i.e. u satisfies (in the sense of distributions)

$$\partial_\alpha f_{p^i_\alpha}(Du(x)) = 0, \qquad i = 1, \cdots, m. \tag{1.2}$$

A classical result of C.B. Morrey ([Mo]) says that, when f satisfies $(*)$ and $n = 2, m \geq 1$, every minimizer of $I(u)$ of type (1.1) is regular. This is also the case when f satisfies $(*)$ and $n \geq 2, m = 1$ by celebrated results of De Giorgi ([De1]) and Nash ([Na]). The methods used in the proof of De Giorgi and Nash can not be extended to the case $m \geq 2$ as shown by a counterexample of De Giorgi ([De2]). The first example of a nonsmooth minimizer for a smooth strongly convex functional of the type (1.1) was constructed by Nečas in high dimensions (see [Ne]). He considered the function $u\colon \mathbf{R}^n \to \mathbf{R}^{n^2}$ defined by

$$u_{ij} = \frac{x_i x_j}{|x|}, \tag{1.3}$$

and for large n constructed a smooth function f satisfying $(*)$ on $M^{n \times n^2}$ for which u is a minimizer of the corresponding functional I. Later Nečas, Hao and Leonardi ([HLN]) were able to modify this construction and make it work for $n \geq 5$. They used u given by

$$u_{ij} = \frac{x_i x_j}{|x|} - \frac{|x|}{n}\delta_{ij}. \tag{1.4}$$

Important counterexamples to regularity of solutions of elliptic systems which are not of the form (1.2) can be found in [GM] and [NJS]. For a comprehensive treatment of regularity questions we refer the reader to [Gi]. Interesting sufficient conditions for regularity are given in [Ko].

The purpose of this paper is to give a counterexample to regularity of weak solutions of (1.2) in the case $n = 3, m = 5$. We use exactly the same u defined by (1.4) and construct a smooth function f satisfying $(*)$ such that u is a minimizer of I. The main idea of our construction is the following. Let $K = \{\nabla u(x), x \in \Omega\}$ be the set of gradients of u. We find a null Lagrangian L (see Definition 2.1 below) such that

$$\nabla L(X) = \nabla f(X), \qquad \forall X \in K \tag{1.5}$$

for a smooth function f satisfying $(*)$. Then u will satisfy the $Euler - Lagrange$ equation of I automatically. To find the null Lagrangian we use the symmetries of the function u. We will see below that there is, up to a multiplicative factor, a unique quadratic null Lagrangian on $M^{5 \times 3}$ which is invariant under the symmetries of the function u. It turns out that this null Lagrangian satisfies a necessary and sufficient condition for the existence of a strongly convex f satisfying (1.5).

2 Preliminaries

First we introduce some basic facts about null Lagrangians.

Definition 2.1 *(see [Ba1]) $L\colon M^{m\times n} \to \mathbf{R}$ is a null Lagrangian if for each smooth $u\colon \mathbf{R}^n \to \mathbf{R}^m$,*

$$\operatorname{div} \nabla L(\nabla u(x)) = 0. \tag{2.1}$$

We recall the following classical theorem about null Lagrangians (see [Da] or [BCO]).

Proposition 1 *Let $L\colon M^{m\times n} \to \mathbf{R}$, the following conditions are equivalent:*
i) L is a null Lagrangian.
ii) L is a linear combination of subdeterminants.
iii)L is rank-one affine, i.e. $t \to L(A + tB)$ is affine for each $A \in M^{m\times n}$ and each $B \in M^{m\times n}$ with rank $B = 1$.

From now on, let Ω be the unit ball in $\mathbf{R}^3$. Consider $u = (u_{ij}(x))$ given by

$$u_{ij}(x) = \frac{x_i x_j}{|x|} - \frac{|x|}{3}\delta_{ij}, \qquad i, j = 1, ..., 3.$$

Then for each $x \in \Omega, u(x) \in \{A \in M^{3\times 3}, A = A^t, trA = 0\} \cong \mathbf{R}^5$. For each $R \in SO(3)$ we have

$$u(Rx) = Ru(x)R^t = \rho_5(R)u(x),$$

where we denote by ρ_{2i+1} the unique irreducible representation of $SO(3)$ of dimension $2i + 1$. This notation will be used throughout the paper. We remark that these representation are of the real type, and therefore for our purposes we do not have to distinguish between the representations over the real numbers and the complex numbers. An easy calculation shows that

$$\nabla u(Rx) = \rho_5(R)\nabla u(x)R^t = \rho_5 \otimes \rho_3(R)\nabla u(x).$$

Lemma 2.1 *There exists a unique (up to multiplication by a scalar) quadratic null Lagrangian L on $M^{5\times 3}$ which is invariant under the above action of $SO(3)$.*

Proof. Consider the tensor space $T = \{a_{ijk} \in (\mathbf{R}^3)^{\otimes 3} | a_{ijk} = a_{jik}, a_{iik} = 0\}$. Clearly we have $T \cong \mathbf{R}^{15} \cong M^{5\times 3}$. By the Clebsch-Gordan formula (see [BD]), we know that

$$\rho_5 \otimes \rho_3 = \rho_7 \oplus \rho_5 \oplus \rho_3.$$

We now identify the quadratic null Lagrangians on $M^{5\times 3}$ with $\Lambda^2\mathbf{R}^3 \otimes \Lambda^2\mathbf{R}^5 \cong \mathrm{Hom}(\Lambda^2\mathbf{R}^3, \Lambda^2\mathbf{R}^5)$ and consider the representation σ of $SO(3)$

on $\mathrm{Hom}(\Lambda^2\mathbf{R}^3, \Lambda^2\mathbf{R}^5)$ induced by $\rho_3 \otimes \rho_5$. By classical group representation theory (see [BD]) we have

$$\sigma = \rho_9 \oplus \rho_7 \oplus \rho_5 \oplus \rho_5 \oplus \rho_3 \oplus \rho_1.$$

Therefore we see there is a unique one dimensional invariant subspace.

3 Constructions

3.1 Construction of L

Now we calculate explicitly the invariant quadratic null Lagrangian which will be denoted by L in what follows. (We slightly abuse the notation, since L is only determined up to a multiplicative factor.) Since we have $M^{5\times 3} = V_7 \oplus V_5 \oplus V_3$, where V_i is the $i-$dimensional irreducible invariant subspace. We know from the classical invariant theory (see [We1]) that L must be of the following form:

$$L(A) = \alpha|X|^2 + \beta|Y|^2 + \gamma|Z|^2$$

where $A \in M^{5\times 3}, \quad A = X + Y + Z, \quad$ with $X \in V_7, Y \in V_5, Z \in V_3$.

We identify $M^{5\times 3}$ with $T = \{a_{ijk} \in (\mathbf{R}^3)^{\otimes 3} | a_{ijk} = a_{jik}, a_{iik} = 0\}$ in the obvious way. Now we use a classical procedure to decompose T into irreducible subspaces (see [We1]). We first decompose T into the trace-free part T' and its orthogonal supplement T_3, i.e. $T = T' \oplus T_3$. An easy calculation shows that the projection on T_3 is given by $a_{ijk} \to -\frac{1}{5}\delta_{ij}\eta_k + \frac{3}{10}\delta_{ki}\eta_j + \frac{3}{10}\delta_{jk}\eta_i$ with $\eta_k = a_{kii}, k = 1, 2, 3$. Then we decompose T' by using symmetrizations. We have $T' = T_1 \oplus T_2$, where the projection on T_1 is given by symmetrization, i.e. $a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{jki} + a_{kij})$; the projection on T_2 is given by $a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{jik} - a_{kji} - a_{kij})$, which corresponds to the following Young tableau:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

We remark that the antisymmetric part of any tensor in T is 0. We now identify T_1 with V_7, T_2 with V_5, T_3 with V_3.

We now use the condition that L has to vanish on rank-one matrices. These matrices correspond to the tensors in T which are of the form $a_{ijk} = c_{ij}\xi_k$, where $C = (c_{ij})$ is a trace-free symmetric matrix. A direct calculation of the norms of the projections a^l_{ijk} of a_{ijk} to T_l gives

$$a_{ijk} = a^1_{ijk} + a^2_{ijk} + a^3_{ijk},$$

with

$$|a^1_{ijk}|^2 = \frac{1}{3}|C|^2|\xi|^2 + \frac{2}{5}|C\xi|^2, \quad |a^2_{ijk}|^2 = \frac{2}{3}|C|^2|\xi|^2 - |C\xi|^2,$$
$$|a^3_{ijk}|^2 = \frac{3}{5}|C\xi|^2.$$

From this we see that, using the same notation as above, $L(A) = \alpha|X|^2 + \beta|Y|^2 + \gamma|Z|^2$ vanishes on rank one matrices if and only if

$$\alpha : \beta : \gamma = (-2) : 1 : 3.$$

For our purpose, we will take $\alpha = -2, \beta = 1, \gamma = 3$ in the following.

3.2 The construction of f

We recall that $K = \{\nabla u(x), x \in \Omega\} = \{\nabla u(x), x \in S^2\} \subset M^{5\times 3}$, where u is defined by (1.4), and where we have identified the 3×3 trace-free symmetric matrices with $\mathbf{R}^5$. A necessary condition for the existence of a strongly convex function f satisfying (1.5) is that there exist $\delta_0 > 0$, such that

$$\nabla L(X) \cdot (Y - X) \leq -\delta_0 |Y - X|^2 \qquad \forall X, Y \in K. \tag{3.1}$$

We will see this condition is satisfied.

Lemma 3.1 *For any $X = \nabla u(x), Y = \nabla u(y) \in K$, where $x, y \in S^2$, we have*

$$L(\nabla u(x) - \nabla u(y)) \geq 8|x - y|^2.$$

Proof. First we note that we have the following decomposition for $\nabla u(x) \in K, x \in S^2$.

$$u_{ijk} = u^1_{ijk} + u^2_{ijk} + u^3_{ijk},$$

where

$$u^1_{ijk} = -x_i x_j x_k + \frac{1}{5}(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}),$$
$$u^2_{ijk} = 0,$$
$$u^3_{ijk} = \frac{4}{5}(x_i\delta_{jk} + x_j\delta_{ik} - \frac{2}{3}x_k\delta_{ij}).$$

and

$$|u^1_{ijk}|^2 = \frac{2}{5}, \quad |u^3_{ijk}|^2 = \frac{64}{15}.$$

Hence $$L(\nabla u(x)) \equiv 12 \qquad \forall x \in S^2.$$

Since L is quadratic, we have

$$L(\nabla u(x) - \nabla u(y)) = 2L(\nabla u(x)) - 2L(\nabla u(x), \nabla u(y)),$$

where we slightly abuse the notation by using L also for the symmetric bilinear form corresponding to the quadratic form L.

$$\begin{aligned}
L(\nabla u(x), \nabla u(y)) &= -2u^1_{ijk}(x) \cdot u^1_{ijk}(y) + 3u^3_{ijk}(x) \cdot u^3_{ijk}(y) \\
&= -2\left(-x_i x_j x_k + \frac{1}{5}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})\right) \cdot \\
&\qquad \left(-y_i y_j y_k + \frac{1}{5}(y_i\delta_{jk} + y_j\delta_{ik} + y_k\delta_{ij})\right) \\
&\qquad +3\left(\frac{4}{5}\right)^2 (x_i\delta_{jk} + x_j\delta_{ki} - \frac{2}{3}x_k\delta_{ij}) \\
&\qquad\qquad \cdot(y_i\delta_{jk} + y_j\delta_{ik} - \frac{2}{3}y_k\delta_{ij}) \\
&= -2\langle x, y\rangle^3 + 14\langle x, y\rangle.
\end{aligned}$$

Let $t = \langle x, y\rangle$. Then $-1 \le t \le 1$, and we have

$$\begin{aligned}
L(\nabla u(x) - \nabla u(y)) &= 2L(\nabla u(x)) - 2L(\nabla u(x), \nabla u(y)) \\
&= 2(1-t)\left(-2(1+t+t^2) + 14\right) \\
&\ge 16(1-t) \\
&= 8|x-y|^2.
\end{aligned}$$

The proof of Lemma 3.1 is finished.

We have $L(X) = 12$ for all $X \in K$ and therefore Lemma 3.1 gives

$$\begin{aligned}
&\nabla L(\nabla u(x)) \cdot (\nabla u(y) - \nabla u(x)) \\
&\quad = -L(\nabla u(x) - \nabla u(y)) \\
&\qquad +L(\nabla u(x)) + L(\nabla u(y)) - 2L(\nabla u(x)) \\
&\quad = -L(\nabla u(x) - \nabla u(y)) \\
&\quad \le -8|x-y|^2.
\end{aligned}$$

Since we have

$$\frac{53}{12}|x-y|^2 \le |X-Y|^2 \le \frac{20}{3}|x-y|^2$$

for $X = \nabla u(x), Y = \nabla u(y)$, we see that the condition (3.1) is satisfied.

It turns out that (3.1) together with the fact that L is constant on K is also sufficient for the existence of a strongly convex function satisfying (1.5). A natural attempt to make such an extension would be to take the convex hull of K and consider a modification of the corresponding Minkowski functional. However, since the convex hull of K may not be smooth at K, we need to slightly modify this construction.

We fix $\epsilon > 0$ (the exact value will be specified later) and for each $X \in K$, consider the 10 dimensional ball of radius $r_\epsilon = \epsilon|\nabla L(X)| = \epsilon\sqrt{160}$ passing through X centered at $X' = X - \nabla L(X)\epsilon$. We will denote the ball as B_{X',r_ϵ}.

Lemma 3.2 *When ϵ is sufficiently small we have*

$$\nabla L(X)(\tilde{Y} - X) \le -\frac{1}{2}|\tilde{Y} - X|^2, \tag{3.2}$$

for each $X \in K$ and each $\tilde{Y} \in B_{Y',r_\epsilon}$, where B_{Y',r_ϵ} is defined above, with Y being an arbitrary point of K.

Proof. The inequality

$$|\tilde{Y} - Y'|^2 \le \epsilon^2|\nabla L(Y)|^2$$

gives

$$\nabla L(Y) \cdot (\tilde{Y} - Y) \le -\frac{1}{2\epsilon}|\tilde{Y} - Y|^2$$

Hence

$$\begin{aligned}\nabla L(X) \cdot (\tilde{Y} - X) &= (\nabla L(X) - \nabla L(Y)) \cdot (\tilde{Y} - Y) \\ &\quad + \nabla L(Y) \cdot (\tilde{Y} - Y) + \nabla L(X) \cdot (Y - X) \\ &\le 10|Y - X||\tilde{Y} - Y| - \frac{1}{2\epsilon}|\tilde{Y} - Y|^2 - \frac{6}{5}|Y - X|^2,\end{aligned}$$

and the statement follows easily.

Let $S = \cup_{X \in K} B_{X',r_\epsilon}$. When ϵ is small, the boundary of S is smooth by elementary results about tubular neighborhoods (see [Hi] or [We2]). Lemma 3.2 implies that (for sufficiently small ϵ) all the eigenvalues of the second fundamental form of ∂S are negative and bounded above uniformly on K by a negative constant γ (i.e. the principle curvatures $k_i(X) \le \gamma < 0, \forall i$ and $\forall X \in K$). Since ∂S is smooth, we conclude that ∂S is locally strongly convex at any point of $U \cap \partial S$, where U is a small neighborhood of K.

Now take G to be the convex hull of S in $V_7 \oplus V_3$. Using Lemma 3.2 and the fact that ∂S is smooth and locally strongly convex in $U \cap \partial S$, we infer that $U \cap \partial G = U \cap \partial S$ when the neighborhood U of K is chosen to be sufficiently small. Let

$$F_1(X) = \min\{t \ge 0, X \in tG\}, \quad F(X) = 12F_1^2(X).$$

Then F is smooth and strongly convex in U (see [Ro]), and $\nabla L(X) = \nabla F(X)$ for each $X \in K$. Let ϕ be a smooth non-negative mollifier with support in B_1 and let

$$F_\delta = \phi_\delta * F,$$

where $\phi_\delta(x) = \delta^{-n}\phi(\frac{x}{\delta})$. Define

$$H_{\delta,\tau}(X) = F_\delta + \tau|X|^2.$$

Let $0 \leq \eta \leq 1$ be a smooth cut-off function satisfying $\eta = 1$ in U', and $\eta = 0$ outside U, where U' is an open neighborhood of K satisfying $\bar{U}' \subset U$.

Now define

$$H = (1-\eta)H_{\delta,\tau} + \eta F.$$

A straightforward calculation shows that H is a strongly convex function satisfying $(*)$ on $V_7 \oplus V_3$ when δ and τ are small enough. Now take $f(A) = H(X+Y) + |Z|^2$ to be our final function, where $A \in M^{5\times 3}$, $A = X+Y+Z$, with $X \in V_7, Y \in V_5, Z \in V_3$. We know that f coincide with $F(X+Y) + |Z|^2$ in a neighborhood of K, thus $\nabla L(X) = \nabla f(X)$ for all $X \in K$ holds and f is a smooth function satisfying $(*)$. This proves the following theorem:

Theorem 1 *Let $\Omega = \{x \in \mathbf{R}^3, |x| < 1\}$ and let u: $\Omega \to \mathbf{R}^5$ be defined by $u_{ij} = \frac{x_i x_j}{|x|} - \frac{|x|}{3}\delta_{ij}$, $i, j = 1, \cdots 3$, where we identify the 3×3 symmetric trace-free matrices with $\mathbf{R}^5$. Then u is a minimizer of $I(u) = \int_\Omega f(Du(x))$, where f is the smooth strongly convex function defined above.*

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