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LAYER SOLUTIONS FOR A ONE-DIMENSIONAL NONLOCAL MODEL OF GINZBURG–LANDAU TYPE

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Abstract. We study a nonlocal model of Ginzburg–Landau type that gives rise to an equation involving a mixture of the Laplacian and half-Laplacian. Our focus is on one-dimensional transition layer profiles that connect the two distinct homogeneous phases. We first introduce a renormalized one-dimensional energy that is free from a logarithmic divergence due to the failure of the Gagliardo norm to be finite on smooth profiles that asymptote to different limits at infinity. We then prove existence, uniqueness, monotonicity and regularity of minimizers in a suitable class. Lastly, we consider the singular limit in which the coefficient in front of the Laplacian vanishes and prove convergence of the obtained minimizer to the solutions of the fractional Allen–Cahn equation.

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1. Introduction

There has recently been a growing interest in nonlinear partial differential equations (PDEs) that involve nonlocal operators in the form of fractional Laplacian (see, *e.g.*, [8–11, 25–27, 37]). These problems are interesting both mathematically as basic generalizations of the classical semilinear PDEs and physically, since they arise in various modeling contexts. Physically, the nonlocality may arise when the standard diffusion process due to Brownian motion is replaced with anomalous diffusion mediated by Levý flights. It also arises when the underlying model involves components that occupy the spatial compartments of different dimensionality, with a lower-dimensional compartment coupled to the higher-dimensional compartment *via* a boundary condition for an elliptic equation governing the latter. Very often, such a coupling results in an appearance of half-Laplacian in a one- or two-dimensional nonlinear PDE.

In this paper, we consider the following nonlocal semilinear elliptic PDE:

$$-\alpha \Delta u + \beta \left(-\Delta\right)^s u - f\left(u\right) = 0 \qquad x \in \mathbb{R}^n. \tag{1.1}$$

Here $\alpha, \beta > 0$, $0 < s < 1$, $f(u) = -W'(u)$, where $W \in C^2\left(\mathbb{R}\right)$ is a nonnegative double-well potential that vanishes only at $u = \pm 1$, and $W''\left(\pm 1\right) > 0$. For $u \in C^2_{loc}\left(\mathbb{R}^n\right) \cap L^\infty\left(\mathbb{R}^n\right)$ the operator $\left(-\Delta\right)^s$ is the fractional

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Laplacian defined by

$$(-\Delta)^s u(x) := C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\mathrm{d}y = C_{n,s}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\mathrm{d}y,$$

where $C_{n,s}$ is a normalizing constant to guarantee that the symbol of the resulting operator is $|\xi|^{2s}$, see *e.g.* [8], Section 3 for more details.

Equation (1.1) with $s=\frac{1}{2}$ arises, for example, in a biological context as a simple model of the cell polarity patterning network in a developing *Caenorhabditis elegans* embryo [22]. In this problem, signaling molecules exist in a cell as free ligands diffusing through the cytosol, as well as in a membrane-bound form. If an open set $\Omega\subset\mathbb{R}^3$ with smooth boundary represents the cytoplasm and $\Gamma=\partial\Omega$ represents the cell membrane, we can define $u:\partial\Omega\to\mathbb{R}$ to be the surface density of the membrane-bound ligand and $v:\Omega\to\mathbb{R}$ its concentration in the cytosol. Then a highly simplified model of cell signaling takes, after a suitable non-dimensionalization, the form of the following boundary value problem (in the spirit of [17, 28, 29]):

$$\frac{\partial u}{\partial t} = \alpha\Delta_\Gamma u + f(u) - \frac{\partial v}{\partial n} \quad \text{on } \Gamma, \tag{1.2}$$

$$\frac{\partial v}{\partial t} = \Delta v \text{ in } \Omega \quad\text{and}\quad v=\beta u \text{ on } \Gamma. \tag{1.3}$$

Here Δ_Γ is the Laplace–Beltrami operator on Γ, $\partial/\partial n$ is the derivative in the direction of the outward normal to $\partial\Omega$, $f(u)$ is an autocatalytic production rate of the ligand (*e.g.*, from an inactive precursor form), and the relationship between u and v on Γ represents the binding/dissociation quasi-equilibrium at the cell membrane. If the characteristic spatial scale of variation of u is much smaller than the cell size, then zooming in on the region of interest amounts to approximating Ω with a half-space and setting $\Gamma=\mathbb{R}^2$. In that case the stationary solution for v is given by a harmonic function with trace βu on Γ and, therefore, $\partial v/\partial n = \beta(-\Delta)^{1/2}u$ [11]. Substituting this expression into the equation for u then yields (1.1) with $s=\frac{1}{2}$ in $\mathbb{R}^2$.

When $\beta=0$, equation (1.1) reduces to the classical Allen–Cahn equation

$$-\alpha\Delta u + W'(u) = 0. \tag{1.4}$$

Starting from the De Giorgi conjecture of 1978, bounded solutions of (1.4) which are monotone in one direction have attracted a lot of attention over the years. De Giorgi conjecture states the following: when $W'(u)=u^3-u$, if u is an entire smooth solution of (1.4) satisfying $\frac{\partial u}{\partial x_n}>0$ in $\mathbb{R}^n$, then the level sets of u are hyperplanes at least if $n\leq 8$. This conjecture was proved by Ghoussoub and Gui [20] when $n=2$, and by Ambrosio and Cabré [2] when $n=3$. Under the additional assumption of anti-symmetry of solutions, Ghoussoub and Gui [21] established the De Giorgi conjecture for $n=4,5$. Further developments on the conjecture can be found in [3]. Under additional assumption $\lim_{x_n\to\pm\infty}u(x)=\pm1$, De Giorgi conjecture was completely solved by Savin [31, 32] for $4\leq n\leq 8$. Later a counter-example in dimensions $n\geq 9$ was established by Del Pino *et al.* [13].

When $\alpha=0$, the problem becomes the fractional Allen–Cahn equation

$$\beta(-\Delta)^s u + W'(u) = 0.$$

Recently, many efforts have been made to extend De Giorgi's conjecture to the fractional Laplacian case. Existence, uniqueness, symmetry and variational properties as well as asymptotic behavior of layer solutions (bounded monotone in one direction) were first derived in [10] in the physically important case $s=\frac{1}{2}$. Properties of solutions of

$$(-\Delta)^s u - f(u) = 0 \tag{1.5}$$

for a general nonlinearity $f(u)$ are discussed in [8, 9, 36]. The fractional De Giorgi conjecture was proved in [9, 10, 36] for the case $n = 2, s \in (0,1)$, and in [5, 6] for $n = 3$ and $s \geq \frac{1}{2}$. Under additional limit conditions, fractional De Giorgi conjecture was proved for $n = 3$ and $s \in \left(0, \frac{1}{2}\right)$ by Dipierro *et al.* in [16] and by Savin in [33] for $4 \leq n \leq 8$ and $s \in \left(\frac{1}{2}, 1\right)$. The limit condition is removed in [15] for $n = 3$ and $s \in \left(0, \frac{1}{2}\right)$. Most recently, Figalli and Serra [18] proved that when $s = \frac{1}{2}$, every bounded stable solution to (1.5) in $\mathbb{R}^3$ is a one-dimensional profile. By Savin's result [33], stable solutions to (1.5) in $\mathbb{R}^{n-1}$ being one-dimensional implies De Giorgi conjecture for fractional Laplacian in $\mathbb{R}^n$. Using this implication and the announcement made in [33] for $s = \frac{1}{2}$, Figalli and Serra solved the De Giorgi conjecture for half-Laplacian when $n = 4$ (remarkably, such a result is *not* known for the classical case $s = 1$). In addition to the works on fractional Laplacian mentioned above, there are also related works in the literature dealing with different translation-invariant integral operators. For instance, De Giorgi conjecture-type results can be found in [24], and similar results were discussed for different operators in [12].

The methods used in [8–10, 36] are largely based on the equivalence of (1.5) to the nonlinear boundary value problem [11]

$$\begin{cases} \operatorname{div}\left(y^a \nabla u\right) = 0 \quad \text{in } \mathbb{R}_+^{n+1}, \\ (1+a)\, \frac{\partial u}{\partial \nu^a} = W'(u) \quad \text{on } \partial\mathbb{R}_+^{n+1}, \end{cases}$$

where $a = 1 - 2s$, $\mathbb{R}_+^{n+1} = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}$ and

$$\frac{\partial u}{\partial \nu^a} = \lim_{y \to 0} y^a \partial_y u$$

is the generalized exterior normal derivative of u. This extension approach has recently been generalized to study layer solutions of Allen–Cahn type equations in the form of a sum of fractional Laplacians of different orders [7]

$$\sum_{i=1}^{k} \mu_i (-\Delta)^{s_i} u + W'(u) = 0 \text{ in } \mathbb{R}^n,$$

where regularity, sharp energy estimates and 1-D symmetry for monotone solutions have been established.

A different approach to the study of layer solutions of (1.5) is established in [27]. It relates the solutions of (1.5) to local minimizers of the energy functional

$$F(u) = \int_\Omega W(u)\, \mathrm{d}x + \int_\Omega \int_\Omega \frac{\left(u(x) - u(y)\right)^2}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y + 2\int_\Omega \int_{\Omega^c} \frac{\left(u(x) - u(y)\right)^2}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y.$$

A Γ-convergence theory for such energy has been developed in [34, 35]. While the work in [27] connects solutions of (1.5) to local minimizers of a variational integral, since the nonlocal term in the energy functional essentially omits the contributions of u in $\Omega^c \times \Omega^c$, it is not clear that the solution is a global minimizer of an energy functional subject to non-compact perturbations. In the literature, one can also find a discussion on the corresponding parabolic equation

$$u_t + (-\Delta)^s u + W'(u) = \sigma$$

in the presence of forcing $\sigma > 0$. For instance, existence of viscosity solutions can be found in Section 4.2 of [4].

Motivated by the work in [27], we propose to study the following nonlocal model in dimension one. Given $\eta(x) \in C^\infty(\mathbb{R})$ satisfying $|\eta| \leq 1$, $\eta(x) = 1$ for $x \geq 1$, $\eta = -1$ for $x \leq -1$, we consider the following *renormalized* nonlocal Ginzburg–Landau energy

$$\begin{aligned} E_\varepsilon(u) = &\int_{\mathbb{R}} \varepsilon^2 |u'|^2 \,\mathrm{d}x + \int_{\mathbb{R}} W(u)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{(u(x)-u(y))^2}{(x-y)^2} - \frac{(\eta(x)-\eta(y))^2}{(x-y)^2} \right] \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{1.6}$$

The second term in the integrand of the double integral above is needed to eliminate the divergence of that integral at infinity on transition layer profiles. At the same time, at least formally the Euler–Lagrange equation for the energy in (1.6) coincides with the one-dimensional version of (1.1) with $\alpha = 2\varepsilon^2$ and $\beta = 2\pi$. Note that there is no loss of generality in this particular choice of the coefficients.

The main goal of this paper is to relate solutions of (1.1) to global minimizers of an energy functional without prescribing the behavior of u outside a bounded region. In particular, when $\varepsilon \to 0$, whether we can recover a global minimizer of the nonlocal energy functional

$$E_0(u) = \int_{\mathbb{R}} W(u)\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{(u(x)-u(y))^2}{(x-y)^2} - \frac{(\eta(x)-\eta(y))^2}{(x-y)^2} \right] \mathrm{d}x\mathrm{d}y$$

is an open question.

To start our discussion, we first consider the minimization problem for each fixed $\varepsilon > 0$. Let

$$\mathcal{A} = \left\{ u \in H^1_{loc}(\mathbb{R}) \,:\, u - \eta \in H^1(\mathbb{R}) \right\}.$$

By the assumptions on η, it follows immediately that if $u \in \mathcal{A}$ is the precise representative, then

$$\lim_{x\to\infty} u(x) = 1, \quad \lim_{x\to-\infty} u(x) = -1. \tag{1.7}$$

We will show that E_ε attains a unique minimum on $\mathcal{A}$ (up to translations). Moreover, the minimizer is monotone and smooth and satisfies the underlying Euler–Lagrange equation

$$-2\varepsilon^2 u_\varepsilon'' + W'(u_\varepsilon) + 2\pi \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{\frac{1}{2}} u_\varepsilon = 0. \tag{1.8}$$

To prove existence of minimizers, we observe that for any $u \in \mathcal{A}$ we can find a sequence u_n such that $u_n - \eta \in C_0^\infty(\mathbb{R})$ and $E_\varepsilon(u_n)$ approximates $E_\varepsilon(u)$. From this, we can always pick our minimizing sequence v_n such that $v_n - \eta \in C_0^\infty(\mathbb{R})$. Since $v_n - \eta$ is compactly supported, we can apply the same arguments as in [27] to show that we can replace v_n by a new minimizing sequence consisting of nondecreasing functions. For such a sequence, we can find a uniform bound in $H^1(\mathbb{R})$, which thus yields a limit function that attains the minimum of $E_\varepsilon(u)$ in $\mathcal{A}$. The regularity proof then follows from a careful analysis for the nonlocal term and a bootstrap argument.

To state our results rigorously, let $E(u) = E_1(u)$. Our first main result is the following existence and regularity theorem.

Theorem 1.1. *Assume that the potential $W \in C^2(\mathbb{R})$ satisfies $W(u) > 0$ for all $u \in (-1,1)$,*

$$W'(-1) = W'(1) = 0 \quad \textit{and} \quad W(-1) = W(1) = 0, \tag{1.9}$$

and $W''(\pm 1) > 0$. *Then there exists a unique (up to translations) minimizer* u_0 *of* $E(u)$ *in* $\mathcal{A}$ *such that* $|u_0| \leq 1$. *Moreover,* $u_0 \in C^{2,\frac{1}{2}}(\mathbb{R})$ *is monotone increasing, satisfies the Euler–Lagrange equation*

$$-2u_0'' + W'(u_0) + 2\pi \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} u_0 = 0, \tag{1.10}$$

and $u_0'(x), u_0''(x) \to 0$ *as* $x \to \pm\infty$.

Here we understand the fractional operator in the following sense:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} u_0(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y| \geq \varepsilon} \frac{u_0(x) - u_0(y)}{(x-y)^2} \, \mathrm{d}y.$$

Note that assumption (1.9) is a necessary and sufficient condition for the existence of an increasing solution of (1.1) when $\alpha = 0$ or $\beta = 0$ (see *e.g.* [10]). For general α and β, the necessity of condition (1.9) can also be seen *via* integration by parts (see Appendix A).

By the same argument as in the proof of Theorem 1.1, we can find a minimizer u_ε of E_ε in $\mathcal{A}$ such that u_ε is monotone increasing, $|u_\varepsilon| \leq 1$, $u_\varepsilon - \eta \in H^1(\mathbb{R})$ and $u_\varepsilon \in C^{2,\frac{1}{2}}(\mathbb{R})$ satisfies (1.8). We establish the following theorem regarding the behavior of u_ε as $\varepsilon \to 0$.

Theorem 1.2. *Let* u_ε *be the unique minimizer of* E_ε *over* $\mathcal{A}$ *with* $u_\varepsilon(0) = 0$. *Then*

1. *There exists a constant* $C > 0$ *such that* $\int_{\mathbb{R}} \varepsilon^2 |u_\varepsilon'|^2 \, \mathrm{d}x + \|u_\varepsilon - \eta\|_{L^2(\mathbb{R})} + \|u_\varepsilon - \eta\|_{H^{1/2}(\mathbb{R})} \leq C$ *for every* ε *sufficiently small.*
2. *For every sequence of* $\varepsilon \to 0$ *there exists a subsequence* u_{ε_j} *such that* $v_{\varepsilon_j} = u_{\varepsilon_j} - \eta$ *converges strongly to* $v \in L^q_{loc}(\mathbb{R})$ *for any* $q \geq 2$, *and* $v_{\varepsilon_j} \rightharpoonup v$ *in* $H^{\frac{1}{2}}(\mathbb{R})$.
   $u = v + \eta$ *satisfies* $u(0) = 0$ *and is the unique monotone nondecreasing weak solution of*

$$W'(u) + 2\pi \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} u = 0, \tag{1.11}$$

   and $u \in C^{2,\frac{1}{2}}(\mathbb{R})$.
3. u *is a minimizer of* E_0 *in* $\mathcal{A}$.

At first it seems that our set of admissible functions for $E_0(u)$ is slightly different from the one considered in [27], where the authors consider admissible functions $u \in L^1_{loc}(\mathbb{R})$ with $\lim_{x\to\pm\infty} u(x) = \pm 1$. Our set of admissible functions comes naturally from our method where we prove minimality of u through minimality of u_ε for $E_\varepsilon(u)$. In our setting, the global minimizer defined in [27] corresponds to minimization with respect to compact perturbations in $L^1_{loc}(\mathbb{R})$. Recall that the minimizer obtained in [27] is monotone increasing and satisfies (1.11). It then follows by uniqueness (up to translation) of monotone nondecreasing weak solution of (1.11) that u indeed is the minimizer obtained in [27]. In other words, we recovered the *local* minimizer in [27] as a *global* minimizer (with respect to all perturbations) of a variational integral defined on the whole domain.

In this paper we focus on the case $s = \frac{1}{2}$. For $s \neq \frac{1}{2}$, recall that by the results in [27] the minimum value of the truncated energy

$$F(u, I) = \int_I W(u) \, \mathrm{d}x + \int_I \int_I \frac{(u(x) - u(y))^2}{|x-y|^{2s+1}} + 2 \int_I \int_{I^c} \frac{(u(x) - u(y))^2}{|x-y|^{2s+1}}$$

on $I = [-R, R]$ grows algebraically as R^{1-2s} for $s < \frac{1}{2}$, logarithmically when $s = \frac{1}{2}$, and bounded above for $s > \frac{1}{2}$. To study global minimizers of a variational integral on the whole domain, it is therefore not necessary to renormalize the energy in the case $s > \frac{1}{2}$. For $s < \frac{1}{2}$, one could consider the following renormalized energy

$$\begin{aligned} E_\varepsilon(u) = &\int_{\mathbb{R}} \varepsilon^2 |u'|^2 \, \mathrm{d}x + \int_{\mathbb{R}} W(u) \, \mathrm{d}x \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{(u(x) - u(y))^2}{|x-y|^{2s+1}} - \frac{(\eta(x) - \eta(y))}{|x-y|^{2s+1}} \right] \mathrm{d}x\mathrm{d}y \end{aligned}$$

and try to apply a similar argument as in present paper.

Finally, turning back to our original problem in (1.1) in the physical case of $s = \frac{1}{2}$ and $n = 2$, in the spirit of De Giorgi's conjecture it is natural to ask whether monotone or locally minimizing (in the sense of [27]) solutions of (1.1) are one-dimensional and, hence, minimizers of the corresponding one-dimensional energy. In particular, it is interesting whether the one-dimensional minimizing property of layer solutions can be useful in establishing rigidity of multi-dimensional solutions of (1.1). A closely related question is the behavior of solutions of (1.1) on bounded domains when both α and β jointly vanish with an appropriate balance between the two (for a related result, see [34]). These are all interesting questions that will be addressed in the future studies.

2. Proof of the main results

2.1. Existence of a minimizer

We first state the following translation invariant lemma.

Lemma 2.1. *Given any constant c, let $u_c(x) = u(x+c)$, then $E(u_c) = E(u)$.*

Proof. Since the first two terms are translation invariant, we can write $E(u_c) = E(u) + D(\eta_c, \eta)$, where

$$D(\eta_c, \eta) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{(\eta_c(x) - \eta_c(y))^2}{(x-y)^2} - \frac{(\eta(x) - \eta(y))^2}{(x-y)^2} \right) \mathrm{d}x\mathrm{d}y.$$

Note that $D(\eta_{-c}, \eta) = D(\eta, \eta_c) = -D(\eta_c, \eta)$, we may assume $c \geq 0$. From the definitions of η and η_c, we have

$$\begin{aligned} D(\eta_c, \eta) = &\int_{-1-c}^{1} \int_{-1-c}^{1} \frac{(\eta_c(x) - \eta_c(y))^2 - (\eta(x) - \eta(y))^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x \\ &+2 \int_{-1-c}^{1} \int_{-\infty}^{-1-c} \frac{(\eta_c(x)+1)^2 - (\eta(x)+1)^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x \\ &+2 \int_{-1-c}^{1} \int_{1}^{\infty} \frac{(\eta_c(x)-1)^2 - (\eta(x)-1)^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x. \end{aligned} \tag{2.1}$$

The first term of (2.1) can be split into two parts:

$$\begin{aligned} &\int_{-1-c}^{1} \int_{-1-c}^{1} \frac{(\eta_c(x) - \eta_c(y))^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x \\ = &\int_{-1-c}^{1-c} \int_{-1-c}^{1-c} \frac{(\eta_c(x) - \eta_c(y))^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x + 2 \int_{-1-c}^{1-c} \int_{1-c}^{1} \frac{(\eta_c(x)-1)^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x \\ = &\int_{-1}^{1} \int_{-1}^{1} \frac{(\eta(x) - \eta(y))^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x + 2 \int_{-1}^{1} \int_{1}^{1+c} \frac{(\eta(x)-1)^2}{(x-y)^2} \mathrm{d}y\mathrm{d}x, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\int_{-1-c}^{1}\int_{-1-c}^{1}\frac{(\eta(x)-\eta(y))^2}{(x-y)^2}\mathrm{d}y\mathrm{d}x\\
&=\int_{-1}^{1}\int_{-1}^{1}\frac{(\eta(x)-\eta(y))^2}{(x-y)^2}\mathrm{d}y\mathrm{d}x+2\int_{-1}^{1}\int_{-1-c}^{-1}\frac{(\eta(x)+1)^2}{(x-y)^2}\mathrm{d}y\mathrm{d}x.
\end{aligned}\tag{2.3}$$

We estimate the second and the third term in (2.1) as follows,

$$\begin{aligned}
&\int_{-1-c}^{1}\int_{-\infty}^{-1-c}\frac{(\eta_c(x)+1)^2-(\eta(x)+1)^2}{(x-y)^2}\mathrm{d}y\mathrm{d}x\\
&=\int_{-1-c}^{1}\frac{(\eta_c(x)+1)^2-(\eta(x)+1)^2}{x+1+c}\mathrm{d}x\\
&=\int_{-1}^{1}\frac{(\eta(x)+1)^2}{x+1}\mathrm{d}x-\int_{-1}^{1}\frac{(\eta(x)+1)^2}{x+1+c}\mathrm{d}x+\int_{1-c}^{1}\frac{4}{x+1+c}\mathrm{d}x\\
&=\int_{-1}^{1}\frac{(\eta(-x)+1)^2}{1-x}\mathrm{d}x-\int_{-1}^{1}\frac{(\eta(-x)+1)^2}{1-x+c}\mathrm{d}x+\int_{-1-c}^{-1}\frac{4}{1-x}\mathrm{d}x
\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}
&\int_{-1-c}^{1}\int_{1}^{\infty}\frac{(\eta_c(x)-1)^2-(\eta(x)-1)^2}{(x-y)^2}\mathrm{d}y\mathrm{d}x\\
&=\int_{-1-c}^{1}\frac{(\eta_c(x)-1)^2-(\eta(x)-1)^2}{1-x}\mathrm{d}x\\
&=\int_{-1}^{1}\frac{(\eta(x)-1)^2}{1-x+c}\mathrm{d}x-\int_{-1}^{1}\frac{(\eta(x)-1)^2}{1-x}\mathrm{d}x-\int_{-1-c}^{-1}\frac{4}{1-x}\mathrm{d}x.
\end{aligned}\tag{2.5}$$

Combining (2.2)–(2.5) we have

$$\begin{aligned}
\frac{1}{2}D(\eta_c,\eta)&=\int_{-1}^{1}\int_{1}^{1+c}\frac{(\eta(x)-1)^2}{(x-y)^2}\mathrm{d}y\mathrm{d}x-\int_{-1}^{1}\int_{-1-c}^{-1}\frac{(\eta(x)+1)^2}{(x-y)^2}\mathrm{d}y\mathrm{d}x\\
&\quad+\int_{-1}^{1}\frac{(\eta(x)-1)^2-(\eta(-x)+1)^2}{1-x+c}\mathrm{d}x+\int_{-1}^{1}\frac{(\eta(-x)+1)^2-(\eta(x)-1)^2}{1-x}\mathrm{d}x\\
&=\int_{-1}^{1}\frac{(\eta(x)+1)^2-(\eta(-x)-1)^2}{1+x+c}\mathrm{d}x-\int_{-1}^{1}\frac{(\eta(-x)+1)^2-(\eta(x)-1)^2}{1-x}\mathrm{d}x\\
&\quad+\int_{-1}^{1}\frac{(\eta(x)-1)^2-(\eta(-x)+1)^2}{1-x+c}\mathrm{d}x+\int_{-1}^{1}\frac{(\eta(-x)+1)^2-(\eta(x)-1)^2}{1-x}\mathrm{d}x\\
&=0.
\end{aligned}\tag{2.6}$$

The last equal sign comes from a change of variable: $\tilde{x}=-x$ in the third term. □

For the rest of the paper, we introduce the notation

$$B_{I\times J}(u,v):=\int_{I}\int_{J}\frac{(v(x)-v(y))(u(x)-u(y))}{(x-y)^2}\mathrm{d}x\mathrm{d}y.$$

Lemma 2.2. *Given $u \in \mathcal{A}$, there exists a sequence $u_n \in \mathcal{A}$ such that $u_n - \eta \in C_0^\infty(\mathbb{R})$ and $E(u_n) \to E(u)$ as $n \to \infty$.*

Proof. Since $v = u - \eta \in H^1(\mathbb{R})$, by density theorem, we can find a sequence $v_n \in C_0^\infty(\mathbb{R})$ such that $v_n \to v$ in $H^1(\mathbb{R})$. It then follows from the embedding theorem that $v_n \to v$ in $H^{\frac{1}{2}}(\mathbb{R})$, therefore

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v_n(x) - v(x) - v_n(y) + v(y))^2}{(x-y)^2}\mathrm{d}x\mathrm{d}y = 2\pi [v_n - v]^2_{\overset{\circ}{H}^{\frac{1}{2}}(\mathbb{R})} \to 0.$$

This implies

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v_n(x) - v_n(y))^2}{(x-y)^2}\mathrm{d}x\mathrm{d}y = 2\pi [v_n]^2_{\overset{\circ}{H}^{\frac{1}{2}}(\mathbb{R})} \to 2\pi [v]^2_{\overset{\circ}{H}^{\frac{1}{2}}(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2}\mathrm{d}x\mathrm{d}y, \tag{2.7}$$

and for any subset $I, J \subset \mathbb{R}$,

$$\int_I\int_J \frac{(v_n(x) - v(x) - v_n(y) + v(y))^2}{(x-y)^2}\mathrm{d}x\mathrm{d}y \to 0. \tag{2.8}$$

By the definition of $B_{I\times J}(u, v)$, we have

$$B_{[-1,1]\times[-1,1]}(\eta, \eta) + 2B_{[-1,1]\times[-1,1]^c}(\eta, \eta) < \infty,$$

we conclude from (2.8) and the fact that $v_n \to v$ in $L^2(\mathbb{R})$ that

$$\begin{aligned} B_{\mathbb{R}\times\mathbb{R}}(v_n, \eta) &= B_{[-1,1]\times[-1,1]}(v_n, \eta) + 2B_{[-1,1]\times[-1,1]^c}(v_n, \eta) + 2B_{(-\infty-1]\times[1,\infty)}(v_n, \eta) \\ &= B_{[-1,1]\times[-1,1]}(v_n, \eta) + 2B_{[-1,1]\times[-1,1]^c}(v_n, \eta) + 4\int_1^\infty \frac{v_n(x)}{x+1}\mathrm{d}x - 4\int_{-\infty}^{-1} \frac{v_n(y)}{1-y}\mathrm{d}y \\ &\to B_{[-1,1]\times[-1,1]}(v, \eta) + 2B_{[-1,1]\times[-1,1]^c}(v, \eta) + 4\int_1^\infty \frac{v(x)}{x+1}\mathrm{d}x - 4\int_{-\infty}^{-1} \frac{v(y)}{1-y}\mathrm{d}y \\ &= B_{\mathbb{R}\times\mathbb{R}}(v, \eta). \end{aligned} \tag{2.9}$$

Moreover, by the embedding theorem

$$|u|_\infty \le |v|_\infty + 1 \le \|v\|_{H^1(\mathbb{R})} + 1 < \infty,$$

and

$$|u_n|_\infty \le |v_n|_\infty + 1 \le \|v_n\|_{H^1(\mathbb{R})} + 1 < \infty.$$

Thus there exist $\theta(x), \widetilde{\theta}(x) \in (0,1)$ such that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} W(u_n)\,\mathrm{d}x - \int_{\mathbb{R}} W(u)\,\mathrm{d}x\right| \\
\leq\ & \int_{\mathbb{R}} \left|(u-u_n) W'(\eta + (u-\eta)\theta + (1-\theta)(u_n-\eta))\right| \mathrm{d}x \\
\leq\ & \int_{\mathbb{R}} \left|(u-u_n) W''\left(\eta + (u-\eta)\theta\widetilde{\theta} + (1-\theta)\widetilde{\theta}(u_n-\eta)\right)\right| \left|(u-\eta)\theta + (1-\theta)(u_n-\eta)\right| \mathrm{d}x \\
&+ \left|\int_{-1}^{1} \left|(u-u_n) W'(\eta)\right| \mathrm{d}x\right| \\
\leq\ & c\,\|u-u_n\|_{L^2(\mathbb{R})}\left(\|u-\eta\|_{L^2(\mathbb{R})} + \|u_n-\eta\|_{L^2(\mathbb{R})}\right) \\
\to\ & 0.
\end{aligned} \tag{2.10}
$$

Here the second inequality follows from the assumption that $W \in C^{2,1}(\mathbb{R})$ and $|u| < \infty$. Since $\nabla\eta \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |\nabla v_n + \nabla\eta|^2\,\mathrm{d}x \to \int_{\mathbb{R}} |\nabla v + \nabla\eta|^2\,\mathrm{d}x. \tag{2.11}$$

Let $u_n = v_n + \eta$, then $u_n \in \mathcal{A}$ and we conclude from (2.7), (2.9), (2.10) and (2.11) that

$$
\begin{aligned}
& E(u_n) \\
=\ & \int_{\mathbb{R}} |\nabla v_n + \nabla\eta|^2\,\mathrm{d}x + \int_{\mathbb{R}} W(u_n)\,\mathrm{d}x + B_{\mathbb{R}\times\mathbb{R}}(v_n, v_n) + 2B_{\mathbb{R}\times\mathbb{R}}(v_n, \eta) \\
\to\ & \int_{\mathbb{R}} |\nabla v + \nabla\eta|^2\,\mathrm{d}x + \int_{\mathbb{R}} W(u)\,\mathrm{d}x + B_{\mathbb{R}\times\mathbb{R}}(v, v) + 2B_{\mathbb{R}\times\mathbb{R}}(v, \eta) \\
=\ & E(u),
\end{aligned}
$$

which concludes the proof. □

Given a compact set $I = [a,b] \subset \mathbb{R}$, let

$$\mathcal{A}_I := \{u \in \mathcal{A} :\ u = \eta \text{ outside } I\}.$$

Lemma 2.3. *If $a \leq -1$, $b \geq 1$, there exists a non-decreasing function $u_I \in \mathcal{A}_I$ such that $E(u_I) = \inf_{u\in\mathcal{A}_I} E(u)$.*

Proof. Since I is compact,

$$B_{I\times I}(\eta,\eta) + 2B_{I\times I^c}(\eta,\eta) < \infty.$$

Thus $\inf_{u\in\mathcal{A}_I} E(u) > -\infty$. Pick a minimizing sequence $\{u_n\}$. Then we have

$$\|u_n'\|_{L^2(\mathbb{R})} + \left\|1-u_n^2\right\|_{L^2(\mathbb{R})} + B_{I\times I}(u_n,u_n) + 2B_{I\times I^c}(u_n,u_n) \leq C. \tag{2.12}$$

Without loss of generality, we can assume $|u_n| \leq 1$. It follows from (2.12) that $\|u_n-\eta\|_{H^1(\mathbb{R})} = \|u_n-\eta\|_{H^1(I)}$ is bounded. Thus we can find a subsequence and $v \in H_0^1(\mathbb{R})$ which is supported in I such that

$$u_{n_k} - \eta \rightharpoonup v \ \text{ in } H_0^1(\mathbb{R}).$$

In particular,

$$u_{n_k} - \eta \to v \text{ a.e. in } \mathbb{R}.$$

Let $u_I = v + \eta$, then $u_I \in \mathcal{A}_I$. Lower semicontinuity of the gradient term and Fatou's Lemma imply

$$\liminf_{n\to\infty} E(u_n) \geq E(u_I).$$

Given any $\tau > 0$, let $w_I = u_I(x+\tau)$. Define $m(x) = \min(u_I(x), w_I(x))$, $M(x) = \max(u_I(x), w_I(x))$. It then follows from the definition of $\mathcal{A}_I$ that

$$\begin{aligned} M(x) &= -1 \text{ on } (-\infty, a-\tau], \quad M(x) = 1 \text{ on } [b-\tau, \infty); \\ m(x) &= -1 \text{ on } (-\infty, a], \ m(x) = u(x) \text{ on } [b-\tau, \infty). \end{aligned}$$

Repeating the arguments in Lemma 3 and Corollary 3 from [27], we conclude that u_I is monotone increasing. □

Existence part of Theorem 1.1 follows from the following Lemmas.

Lemma 2.4. *We have $\inf_{u\in\mathcal{A}} E(u) > -C$ for some positive constant C.*

Proof. Given any $u \in \mathcal{A}$, by Lemma 2.2 we can find a function v such that $v - \eta \in C_0^\infty(\mathbb{R})$ and

$$E(u) \geq E(v) - 1. \tag{2.13}$$

Since $\operatorname{supp}(u-\eta) = K$ is compact, we can find $I = [a,b] \supseteq K$ such that $a < -1$, $b > 1$. Then $v \in \mathcal{A}_I$, and by Lemma 2.3 we have $E(v) \geq \inf_{\mathcal{A}_I} E(u) = E(u_I)$, where $u_I \in \mathcal{A}_I$ is non-decreasing. Since E is translation invariant, we can assume w.l.o.g. that $u_I(0) = 0$. By monotonicity of u_I, we then have $u_I(x) \geq 0$ on $[0,\infty)$ and $u_I(x) \leq 0$ on $(-\infty, 0]$. Letting $v_I = u_I - \eta$, it follows by Hölder inequality that

$$\begin{aligned} E(u_I) &\geq \int_{\mathbb{R}} W(u_I)\,\mathrm{d}x + B_{\mathbb{R}\times\mathbb{R}}(v_I, v_I) + 2B_{\mathbb{R}\times\mathbb{R}}(v_I, \eta) \\ &\geq \int_{\mathbb{R}} W(u_I)\,\mathrm{d}x + \frac{1}{2} B_{\mathbb{R}\times\mathbb{R}}(v_I, v_I) + 4\int_1^\infty \frac{v_I(x)}{x+1}\mathrm{d}x - 4\int_{-\infty}^{-1} \frac{v_I(y)}{1-y}\mathrm{d}y \\ &\quad -2B_{[-1,1]\times[-1,1]}(\eta,\eta) - 4B_{[-1,1]\times(-\infty,-1]}(\eta,\eta) - 4B_{[-1,1]\times[1,\infty)}(\eta,\eta) \\ &\geq \int_1^\infty v_I^2\mathrm{d}x + \int_{-\infty}^{-1} v_I^2\mathrm{d}x + \frac{1}{2} B_{\mathbb{R}\times\mathbb{R}}(v_I, v_I) + 4\int_1^\infty \frac{v_I(x)}{x+1}\mathrm{d}x - 4\int_{-\infty}^{-1} \frac{v_I(y)}{1-y}\mathrm{d}y - C \\ &\geq -C. \end{aligned}$$

Combining this with (2.13) concludes the proof. □

Lemma 2.5. *There exists a unique (up to translations) minimizer u_0 of $E(u)$ in $\mathcal{A}$ s.t. $|u_0| \leq 1$, $u_0 - \eta \in H^1(\mathbb{R})$. Moreover, u_0 is monotone increasing.*

Proof. By Lemma 2.4, the energy E is bounded from below on $\mathcal{A}$. Let $\{u_n\}$ be a minimizing sequence for $E(u)$ in $\mathcal{A}$. Without loss of generality, we can assume $|u_n| \leq 1$. By Lemma 2.2, we can pick u_n such that $u_n - \eta \in C_0^\infty(\mathbb{R})$. Since $\operatorname{supp}(u_n - \eta) = M_n$ is compact, we can find $I_n = [a_n, b_n] \supseteq M_n$ such that $a_n < -1$, $b_n > 1$. Such $u_n \in \mathcal{A}_{I_n}$ and $E(u_n) \geq \inf_{\mathcal{A}_{I_n}} E(u) = E(u_{I_n})$. Here equality follows from Lemma 2.3. Thus we can replace our minimizing sequence $\{u_n\}$ by the new minimizing sequence $\{u_{I_n}\}$. For simplicity of notations, we still denote the new sequence as $\{u_n\}$. Since each u_n is nondecreasing and $E(u)$ is translation invariant, without loss of generality we can assume $u_n(0) = 0$. Let $v_n = u_n - \eta$. Since $W''(1) > 0$, there exists $\varepsilon_0 > 0$ such that $W''(1+x) > \frac{1}{2}W''(1)$ for $|x| < \varepsilon_0$. By (1.7) , there exists L_n such that $0 \geq v_n(x) \geq -\varepsilon_0$ for $x \geq L_n$ and $-1 \leq v_n(x) \leq -\varepsilon_0$ for $1 \leq x \leq L_n$. Moreover, by assumption on W, we can write

$$\begin{aligned}
&\int_1^\infty W(u_n)\,\mathrm{d}x + 4\int_1^\infty \frac{v_n(x)}{x+1}\mathrm{d}x \\
&= \int_1^{L_n} W(1+v_n)\,\mathrm{d}x + \int_{L_n}^\infty W(1+v_n)\,\mathrm{d}x + 4\int_1^{L_n} \frac{v_n(x)}{x+1}\mathrm{d}x + 4\int_{L_n}^\infty \frac{v_n(x)}{x+1}\mathrm{d}x \\
&\geq \min_{0\leq x\leq 1-\varepsilon_0} W(x)(L_n-1) - 4\ln\frac{L_n+1}{2} + \int_{L_n}^\infty \frac{W''(1+\theta v_n)}{2} v_n^2\mathrm{d}x - 4\left(\int_{L_n}^\infty v_n^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{L_n}^\infty \frac{\mathrm{d}x}{(x+1)^2}\right)^{\frac{1}{2}} \\
&\geq \min_{0\leq x\leq 1-\varepsilon_0} W(x)(L_n-1) - 4\ln\frac{L_n+1}{2} + \frac{1}{8}W''(1)\int_{L_n}^\infty v_n^2\mathrm{d}x - C. \qquad (2.14)
\end{aligned}$$

A similar argument on $(-\infty,-1]$ finds K_n such that

$$\begin{aligned}
&\int_{-\infty}^{-1} W(u_n)\,\mathrm{d}x - 4\int_{-\infty}^{-1} \frac{v_n(y)}{1-y}\mathrm{d}y \qquad (2.15)\\
&\geq \min_{0\geq x\geq -1+\varepsilon_0} W(x)(K_n-1) - 4\ln\frac{K_n+1}{2} + \frac{1}{8}W''(-1)\int_{-\infty}^{-K_n} v_n^2\mathrm{d}x - C.
\end{aligned}$$

By (2.14) and (2.15), we can estimate $E(u_n)$ as

$$\begin{aligned}
C &\geq E(u_n) \\
&= \|u_n'\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} W(u_n)\,\mathrm{d}x + B_{\mathbb{R}\times\mathbb{R}}(v_n,v_n) + 2B_{\mathbb{R}\times\mathbb{R}}(v_n,\eta) \\
&\geq \|u_n'\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} W(u_n) + \frac{1}{2}B_{\mathbb{R}\times\mathbb{R}}(v_n,v_n) + 4\int_1^\infty \frac{v_n(x)}{x+1}\mathrm{d}x - 4\int_{-\infty}^{-1}\frac{v_n(y)}{1-y}\mathrm{d}y \\
&\quad -2B_{[-1,1]\times[-1,1]}(\eta,\eta) - 4B_{[-1,1]\times(-\infty,-1]}(\eta,\eta) - 4B_{[-1,1]\times[1,\infty)}(\eta,\eta) \\
&\geq \|v_n'\|^2_{L^2(\mathbb{R})} + \min_{0\leq x\leq 1-\varepsilon_0} W(x)(L_n-1) - 4\ln\frac{L_n+1}{2} + \frac{1}{8}W''(1)\int_{L_n}^\infty v_n^2\mathrm{d}x \\
&\quad + \min_{0\geq x\geq -1+\varepsilon_0} W(x)(K_n-1) - 4\ln\frac{K_n+1}{2} + \frac{1}{8}W''(-1)\int_{-\infty}^{-K_n} v_n^2\mathrm{d}x + \pi[v_n]^2_{\overset{\circ}{H}{}^{\frac{1}{2}}(\mathbb{R})} - C.
\end{aligned}$$

From this we conclude that L_n, and K_n are bounded. It then follows that v_n is bounded in $H^1(\mathbb{R})$ as well. This implies that there exists $v \in H^1(\mathbb{R})$ such that (up to a subsequence) $v_n \rightharpoonup v$ in $H^1(\mathbb{R})$. In particular, for any compact subset $K \subset \mathbb{R}$ we have $v_n \rightharpoonup v$ in $H^1(K)$. The embedding theorem implies $v_n \to v$ in $L^2(K)$, therefore $v_n \to v$ a.e. in $\mathbb{R}$. By lower semicontinuity of the gradient term and Fatou's Lemma, we conclude

$$\begin{aligned}
&\liminf_{n\to\infty} \|u_n'\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} W(u_n)\,\mathrm{d}x + \frac{1}{2}B_{\mathbb{R}\times\mathbb{R}}(v_n,v_n) \qquad (2.16)\\
&\geq \|u_0'\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} W(u_0)\,\mathrm{d}x + \frac{1}{2}B_{\mathbb{R}\times\mathbb{R}}(v,v),
\end{aligned}$$

where $u_0 = v + \eta$. Thus $u_n \to u_0$ a.e. Since $u_n(x)$ is nondecreasing, satisfying $u_n \geq 0$ for $x \geq 0$ and $u_n(x) \leq 0$ when $x \leq 0$, we conclude that $u_0(x)$ is nondecreasing with $u_0(x) \geq 0$ when $x \geq 0$ and $u_0(x) \leq 0$ when $x \leq 0$.

It then follows from

$$\liminf_{n\to\infty}\int_{\mathbb{R}} W\left(u_n\right)\mathrm{d}x \geq \int_{\mathbb{R}} W\left(u_0\right)\mathrm{d}x$$

that we must have $\lim_{x\to\infty} u_0\left(x\right) = 1$ and $\lim_{x\to-\infty} u_0\left(x\right) = -1.$, *i.e.*, $u_0 \in \mathcal{A}$. Moreover, we estimate the limit of the mixed term as follows:

$$\begin{aligned}
B_{\mathbb{R}\times\mathbb{R}}\left(v_n,\eta\right) &= B_{[-1,1]\times[-1,1]}\left(v_n,\eta\right) + 2B_{[-1,1]\times(-\infty,-1]}\left(v_n,\eta\right)\\
&\quad +2B_{[-1,1]\times[1,\infty)}\left(v_n,\eta\right) + 2B_{(-\infty,-1]\times[1,\infty)}\left(v_n,\eta\right)\\
&= B_{[-1,1]\times[-1,1]}\left(v_n,\eta\right) + 2\int_{-1}^{1}\frac{v_n\left(x\right)\left(\eta\left(x\right)+1\right)}{1+x}\mathrm{d}x\\
&\quad -2\int_{-\infty}^{-1} v_n\left(y\right)\int_{-1}^{1}\frac{\eta\left(x\right)+1}{\left(x-y\right)^2}\mathrm{d}x\mathrm{d}y + 2\int_{-1}^{1}\frac{v_n\left(x\right)\left(\eta\left(x\right)-1\right)}{1-x}\mathrm{d}x\\
&\quad -2\int_{1}^{\infty} v_n\left(y\right)\int_{-1}^{1}\frac{\eta\left(x\right)-1}{\left(x-y\right)^2}\mathrm{d}x\mathrm{d}y + 4\int_{1}^{\infty}\frac{v_n\left(x\right)}{1+x}\mathrm{d}x - 4\int_{-\infty}^{-1}\frac{v_n\left(y\right)}{1-y}\mathrm{d}y.
\end{aligned}\tag{2.17}$$

Given any $\varepsilon > 0$, pick $L > 0$ such that

$$\left(\int_{L}^{\infty}\frac{\mathrm{d}y}{\left(y-1\right)^2}\mathrm{d}y\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{-L}\frac{\mathrm{d}y}{\left(1+y\right)^2}\mathrm{d}y\right)^{\frac{1}{2}} < \varepsilon.$$

Since $v_n \rightharpoonup v$ in $H^{\frac{1}{2}}\left([-L,L]\right)$, $v_n \to v$ in $L^2\left([-L,L]\right)$, we have

$$\begin{aligned}
B_{[-1,1]\times[-1,1]}\left(v_n,\eta\right) &\to B_{[-1,1]\times[-1,1]}\left(v,\eta\right),\\
\int_{-1}^{1}\frac{v_n\left(x\right)\left(\eta\left(x\right)+1\right)}{1+x}\mathrm{d}x &\to \int_{-1}^{1}\frac{v\left(x\right)\left(\eta\left(x\right)+1\right)}{1+x}\mathrm{d}x,\\
\int_{-1}^{1}\frac{v_n\left(x\right)\left(\eta\left(x\right)-1\right)}{1-x}\mathrm{d}x &\to \int_{-1}^{1}\frac{v\left(x\right)\left(\eta\left(x\right)-1\right)}{1-x}\mathrm{d}x,\\
\int_{1}^{L}\frac{v_n\left(x\right)}{1+x}\mathrm{d}x &\to \int_{1}^{L}\frac{v\left(x\right)}{1+x}\mathrm{d}x,\\
\int_{-L}^{-1}\frac{v_n\left(y\right)}{1-y}\mathrm{d}y &\to \int_{-L}^{-1}\frac{v\left(y\right)}{1-y}\mathrm{d}y,\\
\int_{-L}^{-1} v_n\left(y\right)\int_{-1}^{1}\frac{\eta\left(x\right)+1}{\left(x-y\right)^2}\mathrm{d}x\mathrm{d}y &\to \int_{-L}^{-1} v\left(y\right)\int_{-1}^{1}\frac{\eta\left(x\right)+1}{\left(x-y\right)^2}\mathrm{d}x\mathrm{d}y,\\
\int_{1}^{L} v_n\left(y\right)\int_{-1}^{1}\frac{\eta\left(x\right)-1}{\left(x-y\right)^2}\mathrm{d}x\mathrm{d}y &\to \int_{1}^{L} v_n\left(y\right)\int_{-1}^{1}\frac{\eta\left(x\right)-1}{\left(x-y\right)^2}\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

and

$$
\begin{aligned}
&\left|\int_L^\infty \frac{v_n(x)}{1+x}\mathrm{d}x - \int_{-\infty}^{-L} \frac{v_n(y)}{1-y}\mathrm{d}y\right| \\
&\leq \|v_n\|_{L^2(\mathbb{R})}\left(\left(\int_L^\infty \frac{1}{(x+1)^2}\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{-L}\frac{1}{(1-y)^2}\mathrm{d}y\right)^{\frac{1}{2}}\right) \leq C\varepsilon, \\
&\left|\int_{-\infty}^{-L} v_n(y)\int_{-1}^1 \frac{\eta(x)+1}{(x-y)^2}\mathrm{d}x\mathrm{d}y + \int_L^\infty v_n(y)\int_{-1}^1\frac{\eta(x)-1}{(x-y)^2}\mathrm{d}x\mathrm{d}y\right| \\
&\leq \|v_n\|_{L^2(\mathbb{R})}\left(\left(\int_L^\infty \frac{\mathrm{d}y}{(y-1)^2}\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{-L}\frac{\mathrm{d}y}{(1+y)^2}\right)^{\frac{1}{2}}\right) \leq C\varepsilon.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left|\int_L^\infty \frac{v(x)}{1+x}\mathrm{d}x - \int_{-\infty}^{-L} \frac{v(y)}{1-y}\mathrm{d}y\right| \\
&\leq \|v\|_{L^2(\mathbb{R})}\left(\left(\int_L^\infty \frac{\mathrm{d}x}{(x+1)^2}\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{-L}\frac{\mathrm{d}y}{(1-y)^2}\right)^{\frac{1}{2}}\right) \leq C\varepsilon, \\
&\left|\int_{-\infty}^{-L} v(y)\,\mathrm{d}y\int_{-1}^1 \frac{\eta(x)+1}{(x-y)^2}\mathrm{d}x\mathrm{d}y + \int_L^\infty v(y)\,\mathrm{d}y\int_{-1}^1\frac{\eta(x)-1}{(x-y)^2}\mathrm{d}x\mathrm{d}y\right| \\
&\leq \|v\|_{L^2(\mathbb{R})}\left(\left(\int_L^\infty \frac{\mathrm{d}y}{(y-1)^2}\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{-L}\frac{\mathrm{d}y}{(1+y)^2}\right)^{\frac{1}{2}}\right) \leq C\varepsilon.
\end{aligned}
$$

Letting $n \to \infty$ in (2.17) yields

$$
\begin{aligned}
&\lim_{n\to\infty} B_{\mathbb{R}\times\mathbb{R}}(v_n,\eta) \\
&\geq B_{[-1,1]\times[-1,1]}(v,\eta) + 2\int_{-1}^1 \frac{v(x)(\eta(x)+1)}{1+x}\mathrm{d}x \\
&\quad +2\int_{-1}^1 \frac{v(x)(\eta(x)-1)}{1-x}\mathrm{d}x - 2\int_{-\infty}^{-1} v(y)\int_{-1}^1\frac{\eta(x)+1}{(x-y)^2}\mathrm{d}x\mathrm{d}y \\
&\quad -2\int_1^\infty v(y)\int_{-1}^1 \frac{\eta(x)-1}{(x-y)^2}\mathrm{d}x\mathrm{d}y + 4\int_1^\infty \frac{v(x)}{1+x}\mathrm{d}x - 4\int_{-\infty}^{-1}\frac{v(y)}{1-y}\mathrm{d}y - C\varepsilon \\
&= B_{\mathbb{R}\times\mathbb{R}}(v,\eta) - C\varepsilon.
\end{aligned} \tag{2.18}
$$

Since ε is arbitrary, we conclude from (2.16) and (2.18) that

$$
\liminf_{n\to\infty} E(u_n) \geq E(u_0).
$$

i.e. $u_0 \in \mathcal{A}$ is a minimizer of E.

Lastly, we address the uniqueness of minimizers. Let $u_0, \widetilde{u_0} \in \mathcal{A}$ be two different minimizers of $E(u)$. Following the arguments in Lemma 3 of [27], recall that for any u, v we have

$$\frac{(m(x)-m(y))^2}{(x-y)^2}+\frac{(M(x)-M(y))^2}{(x-y)^2} \le \frac{(u(x)-u(y))^2}{(x-y)^2}+\frac{(v(x)-v(y))^2}{(x-y)^2}, \tag{2.19}$$

where $m(x)=\min(u(x), v(x))$, $M(x)=\max(u(x), v(x))$. Equality holds in (2.19) if and only if

$$(u(x)-v(x))(u(y)-v(y)) \ge 0.$$

Applying (2.19) to $u=u_0,\ v=\widetilde{u_0}$, it then follows

$$E(m(x))+E(M(x)) \le E(u_0(x))+E(\widetilde{u_0}(x)). \tag{2.20}$$

On the other hand, we have $\min(u_0, \widetilde{u_0}) \in \mathcal{A}$ and $\max(u_0, \widetilde{u_0}) \in \mathcal{A}$. Minimality of u_0, $\widetilde{u_0}$ and (2.20) implies

$$E(m(x))+E(M(x)) = E(u_0(x))+E(\widetilde{u_0}(x)),$$

from which we must have

$$u_0(x) \ge \widetilde{u_0}\ (x) \text{ for all } x \in \mathbb{R} \tag{2.21}$$

or

$$u_0(x) \le \widetilde{u_0}\ (x) \quad \text{for all } x \in \mathbb{R}. \tag{2.22}$$

Without loss of generality, we assume (2.21) is true and strict inequality holds at some point. By translation invariance, we can assume $u_0(0)=\widetilde{u_0}\ (0)$ and $u_0(x) > \widetilde{u_0}\ (x)$ for $0 < x \le \tau_0$ for some $\tau_0 > 0$. Pick $\tau \ll 1$ such that

$$u_0(\tau_0-\tau) > \widetilde{u_0}(\tau_0).$$

Consider $\widetilde{v_0}(x)=\widetilde{u_0}(x+\tau)$, then

$$\widetilde{v_0}(0) > u_0(0)$$

while

$$\widetilde{v_0}(\tau_0-\tau) < u_0(\tau_0-\tau).$$

On the other hand, minimality of $u_0, \widetilde{u_0}$ and translation invariance together with (2.19) imply

$$E(\min(u_0, \widetilde{v_0}))+E(\max(u_0, \widetilde{v_0})) = E(u_0)+E(\widetilde{v_0}).$$

Equality implies $u_0(x) \le \widetilde{v_0}(x)$ for all $x \in \mathbb{R}$ or $u_0(x) \ge \widetilde{v_0}(x)$ for all $x \in \mathbb{R}$, a contradiction. $\square$

2.2. Regularity of the minimizer

Proof of the second part of Theorem 1.1.

Proposition 2.6. *The minimizer u_0 is a $C^{2,\frac{1}{2}}(\mathbb{R})$ solution of*

$$-2u_0'' + W'(u_0) + 2\pi\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} u_0 = 0, \tag{2.23}$$

where we understand the fractional operator in the following sense

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} u_0(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon}\frac{u_0(x)-u_0(y)}{(x-y)^2}\,\mathrm{d}y.$$

Proof. Let $v_0 = u_0 - \eta$. We write $E(u_0)$ in terms of v_0 as

$$E(u_0) = \int_{\mathbb{R}} |v_0' + \eta'|^2\,\mathrm{d}x + \int_{\mathbb{R}} W(v_0+\eta)\,\mathrm{d}x + B_{\mathbb{R}\times\mathbb{R}}(v_0, v_0) + 2B_{\mathbb{R}\times\mathbb{R}}(v_0,\eta).$$

Consider now variations $v_\varepsilon = v_0 + \varepsilon\varphi$, where φ is an arbitrary smooth compactly supported function. Since u_0 is a minimizer, we must have

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}E(u_\varepsilon)\right|_{\varepsilon=0} = \int_{\mathbb{R}}(2u_0'\varphi' + W'(u_0)\varphi)\mathrm{d}x + 2B_{\mathbb{R}\times\mathbb{R}}(v_0,\varphi) + 2B_{\mathbb{R}\times\mathbb{R}}(\eta,\varphi). \tag{2.24}$$

Since $v_0 \in H^{\frac{1}{2}}(\mathbb{R})$, we can define $\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} v_0$ *via* Fourier transform as (see *e.g.* [14] Prop. 3.3)

$$\widehat{\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} v_0}(\xi) = |\xi|\,\widehat{v}_0(\xi),$$

and write the second term in (2.24) (see [14] Rem. 3.7) as

$$B_{\mathbb{R}\times\mathbb{R}}(v_0,\varphi) = 2\pi\int_{\mathbb{R}}\varphi(x)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} v_0(x)\,\mathrm{d}x.$$

Since $\eta \in C^\infty(\mathbb{R})$, for $x > 1$, taking $\varepsilon \ll 1$, we have

$$\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y = \int_{-\infty}^{1}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y \leq \frac{2}{x-1}, \tag{2.25}$$

and when $x < -1$, taking $\varepsilon \ll 1$, we have

$$\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y = \int_{-1}^{\infty}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y \leq \frac{2}{x+1}. \tag{2.26}$$

For $-1 \leq x \leq 1$, we can write

$$\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y = \frac{1}{2}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{\eta(x+y)+\eta(x-y)-2\eta(x)}{y^2}\mathrm{d}y = \frac{1}{2}\int_{\mathbb{R}}\frac{\eta(x+y)+\eta(x-y)-2\eta(x)}{y^2}\mathrm{d}y.$$

The last step follows from the fact that $\eta \in C^\infty(\mathbb{R})$, since for each $x \in \mathbb{R}$

$$\left|\int_{\mathbb{R}}\frac{\eta(x+y)+\eta(x-y)-2\eta(x)}{y^2}\mathrm{d}y\right| \leq \int_1^\infty \frac{4}{y^2}\mathrm{d}y + \int_{-\infty}^{-1}\frac{4}{y^2}\mathrm{d}y + \left|\int_{-1}^{1}\frac{\eta(x+y)+\eta(x-y)-2\eta(x)}{y^2}\mathrm{d}y\right| \leq 8 + 2\left\|D^2\eta\right\|_{L^\infty}. \tag{2.27}$$

Combining (2.25), (2.26) and (2.27), we conclude

$$\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y \in L^2(\mathbb{R}). \tag{2.28}$$

Thus the third term in (2.24) can be written as

$$B_{\mathbb{R}\times\mathbb{R}}(\eta,\varphi) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))(\varphi(x)-\varphi(y))}{(x-y)^2}\mathrm{d}x\mathrm{d}y = 2\int_{\mathbb{R}}\varphi(x)\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y\mathrm{d}x.$$

We introduce the notation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u_0 = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_0 + \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y.$$

Since φ is arbitrary, we conclude from (2.24) that u_0 satisfies the following equation in the distributional sense:

$$-2u_0'' + W'(u_0) + 2\pi\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u_0 = 0. \tag{2.29}$$

Since $|u_0| \leq 1$, $W''(\pm 1) > 0$ and $v_0 = u_0 - \eta \in H^1$, we have $W'(u_0) \in L^2(\mathbb{R})$ and $\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_0 \in L^2(\mathbb{R})$. Thus (2.28) implies $\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u_0 \in L^2(\mathbb{R})$. By elliptic estimates, we conclude $u_0 \in H^2(\mathbb{R})$.

Now differentiate (2.29) with respect to x (in weak sense):

$$-2u_{0x}'' + u_{0x}W''(u_0) + 2\pi\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u_{0x} = 0.$$

Here we used the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_0 = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_{0x} \tag{2.30}$$

in the distributional sense and

$$\frac{\mathrm{d}}{\mathrm{d}x}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y=\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{\eta'(x)-\eta'(y)}{(x-y)^2}\mathrm{d}y. \tag{2.31}$$

Equation (2.30) follows *via* Fourier transform. More precisely, by Plancherel formula, we have for any $\phi\in C_0^\infty(\mathbb{R})$:

$$-\int_{\mathbb{R}}\phi'(x)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_0(x)\mathrm{d}x=\int_{\mathbb{R}}i\xi\widehat{\phi^*}(\xi)|\xi|\widehat{v}_0(\xi)\mathrm{d}\xi=\int_{\mathbb{R}}\phi(x)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_{0x}\mathrm{d}x.$$

Equation (2.30) then follows from the definition of weak derivative. Equation (2.31) is a consequence of the following calculation

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y\\
&=\lim_{h\to 0}\frac{1}{h}\left[\lim_{\varepsilon\to 0}\int_{|x+h-y|\geq\varepsilon}\frac{\eta(x+h)-\eta(y)}{(x+h-y)^2}\mathrm{d}y-\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y\right]\\
&=\lim_{h\to 0}\frac{1}{h}\left[\lim_{\varepsilon\to 0}\int_{|x-z|\geq\varepsilon}\frac{\eta(x+h)-\eta(z+h)}{(x-z)^2}\mathrm{d}z-\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y\right]\\
&=\lim_{h\to 0}\frac{1}{h}\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{\eta(x+h)-\eta(x)-\eta(y+h)+\eta(y)}{(x-y)^2}\mathrm{d}y\\
&=\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\lim_{h\to 0}\frac{\eta(x+h)-\eta(x)-\eta(y+h)+\eta(y)}{h(x-y)^2}\mathrm{d}y\\
&=\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{\eta'(x)-\eta'(y)}{(x-y)^2}\mathrm{d}y.
\end{aligned}$$

Following the same arguments as in the proof of

$$\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y\in L^2(\mathbb{R}),$$

we can show that

$$\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{(\eta'(x)-\eta'(y))}{(x-y)^2}\mathrm{d}y\in L^2(\mathbb{R}).$$

Define now

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u_{0x}=\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_{0x}+\frac{\mathrm{d}}{\mathrm{d}x}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon}\frac{(\eta(x)-\eta(y))}{(x-y)^2}\mathrm{d}y.$$

Since $u_{0x}W''(u_0)\in L^2(\mathbb{R})$, $\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}v_{0x}\in L^2(\mathbb{R})$, we have $\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u_{0x}\in L^2(\mathbb{R})$. One can then conclude from the definition of the second weak derivative and the weak formulation of the equation that $u_{0x}\in H^2(\mathbb{R})$. Therefore, $u_0\in H^3(\mathbb{R})\subset C^{2,\frac{1}{2}}(\mathbb{R})$, and u_0 is a classical solution of (2.29). Moreover, since $u_0\in C^{2,\frac{1}{2}}(\mathbb{R})$, we can write

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}} u_0(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon}\frac{(u_0(x)-u_0(y))}{(x-y)^2}\mathrm{d}y.$$

The second part of Theorem 1.1 follows.

Next we give a different proof of uniqueness using Euler–Lagrange equation (2.23) . If there are two monotone increasing minimizers u_1 and u_2, then by minimality and (2.19) we have

$$E(\min(u_1,u_2)) + E(\max(u_1,u_2)) = E(u_1) + E(u_2).$$

Therefore $u_1(x)\geq u_2(x)$ on $\mathbb{R}$ or $u_1(x)\leq u_2(x)$ on $\mathbb{R}$. Assume $u_1(x)\geq u_2(x)$ on $\mathbb{R}$, without loss of generality we may assume $u_i(0)=0$. If $u_1(x)\neq u_2(x)$, then since $u_i\in\mathcal{A}$ this implies u_1-u_2 takes its minimum value 0 at point $x=0$. Applying (2.23) to u_i, we have

$$-2(u_1-u_2)''(0) - 2(u_1-u_2)(0)\left(1-u_1^2(0)-u_1(0)u_2(0)-u_2^2(0)\right) + 2\pi\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}(u_1-u_2)(0) = 0.$$

On the other hand, we have

$$-(u_1-u_2)''(0)\leq 0,\quad (u_1-u_2)(0)=0,$$

and

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}(u_1-u_2)(0) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\varepsilon}\frac{(u_1(0)-u_2(0)-u_1(y)+u_2(y))}{(0-y)^2}\mathrm{d}y < 0,$$

a contradiction. □

2.3. Singular perturbation

Proof of Theorem 1.2. For $\varepsilon_1>\varepsilon_2$, we have

$$E_{\varepsilon_1}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_2}).$$

Thus

$$E_\varepsilon(u_\varepsilon)\leq C.$$

Letting $v_\varepsilon = u_\varepsilon-\eta$, we rewrite $E_\varepsilon(u_\varepsilon)$ in terms of v_ε as follows:

$$E_\varepsilon(v_\varepsilon) = \int_{\mathbb{R}}\left(\varepsilon^2\left|v_\varepsilon'+\eta'\right|^2 + W(v_\varepsilon+\eta)\right)\mathrm{d}x + B_{\mathbb{R}\times\mathbb{R}}(v_\varepsilon,v_\varepsilon) + 2B_{\mathbb{R}\times\mathbb{R}}(v_\varepsilon,\eta).$$

Since u_ε is monotone increasing and $u_\varepsilon(0)=0$, we have $v_\varepsilon(x)\geq -1$ for $x\geq 1$ and $v_\varepsilon(x)\leq 1$ for $x\leq -1$. Applying a similar argument as in the proof of Lemma 2.5, we find L_ε, $K_\varepsilon>0$ such that

$$\begin{aligned}
C &\geq B_{\mathbb{R}\times\mathbb{R}}(v_\varepsilon,v_\varepsilon)+\int_{\mathbb{R}}W(v_\varepsilon+\eta)\,\mathrm{d}x+2B_{\mathbb{R}\times\mathbb{R}}(v_\varepsilon,\eta)\\
&=\int_1^\infty W(v_\varepsilon+\eta)\,\mathrm{d}x+\int_{-\infty}^{-1}W(v_\varepsilon+\eta)\,\mathrm{d}x+\int_{-1}^{1}W(v_\varepsilon+\eta)\,\mathrm{d}x+4B_{(-1,-\infty]\times[1,\infty)}(v_\varepsilon,\eta)\\
&\quad+2B_{[-1,1]\times[-1,1]}(v_\varepsilon,\eta)+4B_{[-1,1]\times(-\infty,-1]}(v_\varepsilon,\eta)+4B_{[-1,1]\times[1,\infty)}(v_\varepsilon,\eta)+B_{\mathbb{R}\times\mathbb{R}}(v_\varepsilon,v_\varepsilon)\\
&\geq \min_{0\leq x\leq 1-\varepsilon_0}W(x)(L_\varepsilon-1)-4\ln\frac{L_\varepsilon+1}{2}+\frac{1}{8}W''(1)\int_{L_\varepsilon}^\infty v_n^2\mathrm{d}x\\
&\quad+\min_{0\geq x\geq -1+\varepsilon_0}W(x)(K_\varepsilon-1)-4\ln\frac{K_\varepsilon+1}{2}+\frac{1}{8}W''(-1)\int_{-\infty}^{-K_\varepsilon}v_n^2\mathrm{d}x-C\\
&\quad+\frac{1}{2}B_{[-1,1]\times[-1,1]}(v_\varepsilon,v_\varepsilon)-2B_{[-1,1]\times[-1,1]}(\eta,\eta)\\
&\quad+B_{[-1,1]\times(-\infty,-1]}(v_\varepsilon,v_\varepsilon)-B_{[-1,1]\times(-\infty,-1]}(\eta,\eta)+B_{(-\infty,-1]\times(-\infty,-1]}(v_\varepsilon,v_\varepsilon)\\
&\quad+B_{[-1,1]\times[1,\infty)}(v_\varepsilon,v_\varepsilon)-B_{[-1,1]\times[1,\infty)}(\eta,\eta)+B_{[1,\infty)\times[1.\infty)}(v_\varepsilon,v_\varepsilon).
\end{aligned}$$

From this, we conclude

$$\int_{\mathbb{R}}\varepsilon^2\left|u_\varepsilon'\right|^2\mathrm{d}x+\|v_\varepsilon\|^2_{L^2(\mathbb{R})}+[v_\varepsilon]^2_{\overset{\circ}{H}^{\frac{1}{2}}(\mathbb{R})}\leq C+E_\varepsilon(v_\varepsilon)\leq C. \tag{2.32}$$

Thus we can find a subsequence $\varepsilon_j\to 0$ and $v\in H^{\frac{1}{2}}(\mathbb{R})$, such that

$$v_{\varepsilon_j}\to v \text{ strongly in } L^2_{loc}(\mathbb{R}) \tag{2.33}$$

and

$$v_{\varepsilon_j}\rightharpoonup v \text{ weakly in } H^{\frac{1}{2}}(\mathbb{R}). \tag{2.34}$$

Since $\left|v_{\varepsilon_j}\right|\leq 2$, we have

$$v_{\varepsilon_j}\to v \text{ strongly in } L^q_{loc}(\mathbb{R}), \tag{2.35}$$

for any $2\leq q<\infty$. Let $u=v+\eta$, then $u_{\varepsilon_j}\to u$ a.e. in $\mathbb{R}$. Thus u is monotone increasing, $|u|\leq 1$ and $\lim_{x\to\pm\infty}u(x)=\pm 1$. Moreover, recalling that u_{ε_j} satisfies (1.8), by (2.32), (2.34) and (2.35), we can pass to the limit in (1.8) and obtain

$$-W'(u)+2\pi\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u=0. \tag{2.36}$$

By the uniqueness and regularity theory [10] of equation (2.36), we conclude that $u\in C^{2,\frac{1}{2}}(\mathbb{R})$ is the unique monotone increasing solution of (2.36). Moreover, the asymptotic behavior of u (see *e.g.* Thm. 1.6 of [11] or Thm. 2 of [27]) states

$$|u(x)-\operatorname{sgn} x|\leq C|x|^{-1},\ |u'(x)|\leq C|x|^{-2} \text{ for } x \text{ large.}$$

This implies that $u-\eta\in H^1(\mathbb{R})$.

Given φ such that $u+\varphi-\eta\in H^1(\mathbb{R})$, $\lim_{x\to\pm\infty}\ (u+\varphi)(x)=\pm1$, then $v_{\varepsilon_j}+\varphi\in H^1(\mathbb{R})$. In particular, $\left|v_{\varepsilon_j}\right|_\infty$, $|v|_\infty<\infty$. By mean value theorem, for any compact domain Ω, the following holds for some $\theta_i(x)\in(0,1)$, $i=1,2,3,4$:

$$\begin{aligned}&\int_{\mathbb{R}} W\left(v_{\varepsilon_j}+\varphi+\eta\right)\mathrm{d}x-\int_{\mathbb{R}} W\left(v_{\varepsilon_j}+\eta\right)\mathrm{d}x-\int_{\mathbb{R}} W(v+\varphi+\eta)\,\mathrm{d}x+\int_{\mathbb{R}} W(v+\eta)\,\mathrm{d}x\\
=&\int_\Omega\left(W'\left(\theta_1 v+(1-\theta_1)v_{\varepsilon_j}+\varphi+\eta\right)\left(v_{\varepsilon_j}-v\right)-W'\left(\theta_2 v_{\varepsilon_j}+(1-\theta_2)v+\eta\right)\left(v_{\varepsilon_j}-v\right)\right)\mathrm{d}x\\
&+\int_{\Omega^c}\left(W'\left(v_{\varepsilon_j}+\theta_3\varphi+\eta\right)\varphi-W'(v+\eta+\theta_4\varphi)\varphi\right)\mathrm{d}x.\end{aligned}$$

Since $\varphi\in L^2(\mathbb{R})$, for every $\varepsilon>0$, there exists compact Ω such that $\int_{\Omega^c}\varphi^2\mathrm{d}x<\varepsilon$. Thus

$$\begin{aligned}&\left|\int_{\Omega^c}\left(W'\left(v_{\varepsilon_j}+\theta_3\varphi+\eta\right)\varphi-W'(v+\eta+\theta_4\varphi)\varphi\right)\mathrm{d}x\right|\\
=&\left|\int_{\Omega^c} W''\left(\theta_5\left(v_{\varepsilon_j}+\theta_3\varphi\right)+(1-\theta_5)(v+\theta_4\varphi)+\eta\right)\varphi\left(v_{\varepsilon_j}+\theta_3\varphi-v-\theta_4\varphi\right)\mathrm{d}x\right|\\
\le& c\left(\int_{\Omega^c}\varphi^2\mathrm{d}x+\left(\int_{\Omega^c}\varphi^2\mathrm{d}x\right)^{\frac12}\left(\int_{\Omega^c}(v_{\varepsilon_j}^2+v^2)\mathrm{d}x\right)^{\frac12}\right)\le C\varepsilon,\end{aligned}$$

and

$$\begin{aligned}&\left|\int_\Omega\left(W'\left(\theta_1 v+(1-\theta_1)v_{\varepsilon_j}+\varphi+\eta\right)\left(v_{\varepsilon_j}-v\right)-W'\left(\theta_2 v_{\varepsilon_j}+(1-\theta_2)v+\eta\right)\left(v_{\varepsilon_j}-v\right)\right)\mathrm{d}x\right|\\
\le& c\left(\int_\Omega\left(v_{\varepsilon_j}-v\right)^2\mathrm{d}x\right)^{\frac12}\to0.\end{aligned}$$

Moreover, since $\varphi\in H^1(\mathbb{R})$, $v_{\varepsilon_j}\rightharpoonup v$ weakly in $H^{\frac12}(\mathbb{R})$, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left(v_{\varepsilon_j}(x)-v_{\varepsilon_j}(y)\right)(\varphi(x)-\varphi(y))}{(x-y)^2}\mathrm{d}x\mathrm{d}y\to\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{(x-y)^2}\mathrm{d}x\mathrm{d}y.$$

Thus

$$\begin{aligned}&E_{\varepsilon_j}\left(v_{\varepsilon_j}+\varphi\right)-E_{\varepsilon_j}\left(v_{\varepsilon_j}\right)\\
=&\int_{\mathbb{R}}\left(\varepsilon_j^2\left|v_{\varepsilon_j}'+\varphi'+\eta'\right|^2+W\left(v_{\varepsilon_j}+\varphi+\eta\right)\right)\mathrm{d}x+B_{\mathbb{R}\times\mathbb{R}}\left(v_{\varepsilon_j}+\varphi,v_{\varepsilon_j}+\varphi\right)+2B_{\mathbb{R}\times\mathbb{R}}\left(v_{\varepsilon_j}+\varphi,\eta\right)\\
&-\int_{\mathbb{R}}\left[\varepsilon_j^2\left|v_{\varepsilon_j}'+\eta'\right|^2+W\left(v_{\varepsilon_j}+\eta\right)\right]\mathrm{d}x-B_{\mathbb{R}\times\mathbb{R}}\left(v_{\varepsilon_j},v_{\varepsilon_j}\right)-2B_{\mathbb{R}\times\mathbb{R}}\left(v_{\varepsilon_j},\eta\right)\\
=&\int_{\mathbb{R}}\varepsilon_j^2|\varphi'|^2\mathrm{d}x+2\int_{\mathbb{R}}\varepsilon_j^2\left(v_{\varepsilon_j}'+\eta'\right)\varphi'\mathrm{d}x+\int_{\mathbb{R}}\left(W\left(v_{\varepsilon_j}+\varphi+\eta\right)-W\left(v_{\varepsilon_j}+\eta\right)\right)\mathrm{d}x\\
&+B_{\mathbb{R}\times\mathbb{R}}(\varphi,\varphi)+2B_{\mathbb{R}\times\mathbb{R}}(\varphi,\eta)+2B_{\mathbb{R}\times\mathbb{R}}\left(v_{\varepsilon_j},\varphi\right)\\
\to&\int_{\mathbb{R}}(W(v+\varphi+\eta)-W(v+\eta))\,\mathrm{d}x+B_{\mathbb{R}\times\mathbb{R}}(\varphi,\varphi)+2B_{\mathbb{R}\times\mathbb{R}}(\varphi,\eta)+2B_{\mathbb{R}\times\mathbb{R}}(v,\varphi)\\
=&E_0(u+\varphi)-E_0(u).\end{aligned}$$

Since

$$E_\varepsilon\left(v_\varepsilon+\varphi\right)-E_\varepsilon\left(v_\varepsilon\right)\geq 0,$$

we conclude that

$$E_0\left(u+\varphi\right)-E_0\left(u\right)\geq 0,$$

i.e., the limit function u is a minimizer of $E_0\left(u\right)$. □

Appendix A

In this appendix, we show the necessity of (1.9) for general $\alpha\neq 0$ and $\beta\in\mathbb{R}$. It is assumed that $f\in C^1(\mathbb{R})$.

Proposition A.7. *Let $u_0\in H^1_{loc}(\mathbb{R})$ be a solution of (1.1) with $n=1$, $s=\frac{1}{2}$ satisfying $|u_0|\leq 1$, $u_0'\in H^1(\mathbb{R})$ and (1.7), and define $W(u)=\int_u^1 f(s)ds$. Then (1.9) holds.*

Proof. First we observe that by our assumptions we have $u_0''\in H^1(\mathbb{R})$ and, in particular, u_0 is a classical solution of (1.1) whose derivatives up to second order vanish at infinity. Now, write

$$\begin{aligned}&\int_{|x-y|\geq\varepsilon}\frac{\left(u_0\left(x\right)-u_0\left(y\right)\right)}{\left(x-y\right)^2}\mathrm{d}y\\
&=\int_{|x-y|\geq L}\frac{\left(u_0\left(x\right)-u_0\left(y\right)\right)}{\left(x-y\right)^2}\mathrm{d}y+\int_{L\geq|x-y|\geq\varepsilon}\frac{\left(u_0\left(x\right)-u_0\left(y\right)\right)}{\left(x-y\right)^2}\mathrm{d}y\\
&=I+II.\end{aligned}$$

For I, we have an estimate

$$|I|\leq 4\int_L^\infty\frac{1}{y^2}\mathrm{d}y=\frac{4}{L}\to 0\quad\text{as } L\to\infty.$$

To estimate II, we can write

$$\begin{aligned}|II|&=\left|\int_{L\geq|x-y|\geq\varepsilon}\frac{\left(u_0\left(x\right)-u_0\left(y\right)\right)}{\left(x-y\right)^2}\mathrm{d}y\right|\\
&=\left|\int_{L\geq|x-y|\geq\varepsilon}\frac{1}{(x-y)^2}\int_y^x\int_x^t u_0''\left(s\right)\mathrm{d}s\mathrm{d}t\mathrm{d}y\right|\\
&\leq CL^{1/2}\left[\int_{x-L}^{x+L}\left|u_0''\left(y\right)\right|^2\mathrm{d}y\right]^{\frac{1}{2}}\to 0\text{ as } |x|\to\infty.\end{aligned}$$

Thus

$$\lim_{x\to\pm\infty}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{\frac{1}{2}}u_0\left(x\right)=0.$$

Letting $x\to\pm\infty$ in (1.1) yields $W'\left(-1\right)=W'\left(1\right)=0$.

We now multiply (1.1) by $u_0' \in H^1(\mathbb{R})$ on both sides and integrate by parts to obtain

$$\begin{aligned}0 &= \int_{\mathbb{R}} \left(-\alpha u_0'' u_0' + W'(u_0) u_0' + \beta u_0' \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{\frac{1}{2}} u_0 \right) \mathrm{d}x \\ &= -\frac{\alpha}{2} (u_0')^2 \Big|_{-\infty}^{\infty} + W(u_0)\big|_{u_0=-1}^{u_0=1} + \beta \int_{\mathbb{R}} u_0' H u_0' \,\mathrm{d}x \\ &= W(1) - W(-1).\end{aligned}$$

Here H is Hilbert transform (see [23] for more details), and we used the property

$$\int_{\mathbb{R}} gHf \mathrm{d}x = -\int_{\mathbb{R}} fHg \mathrm{d}x.$$

This concludes the proof. □

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