A TYPE OF HOMOGENIZATION PROBLEM

Fanghua Lin and Xiaodong Yan

1. Introduction. We consider the following homogenization problem (For relevant discussions, see [BK, L1]). Let Ω be a smooth bounded domain in $\mathbb{R}^d$ with a periodic structure, $\Omega_0$ is a periodic subdomain of $\Omega$ with $|\Omega_0| = \gamma |\Omega|$ for some given constant $\gamma > 0$. $N$ is a smooth compact submanifold of $\mathbb{R}^k$. We consider

$$\min \int_{\Omega} a_{\alpha\beta}(x) \frac{D^\alpha n^\epsilon(x)}{\epsilon} \cdot D^\beta n^\epsilon(x) \, dx$$

subject to, with constants $1 \leq \alpha, \beta \leq d$, $c_1 > 0$, $c_1 + c_2 > 0$,

$$a_{\alpha\beta}(x) = \delta_{\alpha\beta}(c_1 + c_2 \chi_{A_0}) I_k \in M^{k \times k},$$

$$n^\epsilon : \Omega \to N,$$  

$$n^\epsilon |_{\partial \Omega} = g.$$  

Here $M^{k \times k}$ being the set of all $k \times k$ matrices, $I_k$ is the identity matrix on $\mathbb{R}^k$.

The question we are concerned is the regularity for $n^\epsilon$ and the asymptotic behavior as $\epsilon$ tends to zero. The problem can be viewed as an analogue of the usual $\Gamma$ convergence type problem (see for example, [Ms]) onto curved targets. Due to this constraint in the target, we need to apply techniques used for harmonic maps to construct comparison functions in proving the homogenization limit. We follow the ideas in [AL1] to obtain uniform small energy Hölder estimates and Lipschitz estimates. Such uniform Hölder or $W^{1,p}$ estimates were also found in [C] for some different nonlinear homogenization problems using rather different approaches.

The paper is designed as follows. In section 2, we prove partial regularity result of minimizer $n^\epsilon$ for fixed $\epsilon$. We obtain a similar estimates on the size of the singular set as for minimizing harmonic maps. In section 3, we prove the homogenization limit theorem and uniform apriori estimates of $n_\epsilon$ independent of $\epsilon$. We also point out an interesting application of our uniform estimates to obtain a uniform bound on the number of singularities of $n^\epsilon$ in a special case.

2. Regularity of $n^\epsilon$. Let $\Omega$ be a bounded smooth domain of $\mathbb{R}^d$, $A \subset \Omega$ is a smooth subset of $\Omega$ with $|A| = \gamma |\Omega|$, $N$ is a smooth compact submanifold of $\mathbb{R}^k$. We consider the following minimization problem:

$$\min \left\{ \int_{\Omega} a_{\alpha\beta} D^{\alpha} n \cdot D^{\beta} n \, dx, n \in H^1(\Omega, N), n|_{\partial \Omega} = g \right\},$$

where

$$a_{\alpha\beta}(x) = \delta_{\alpha\beta}(1 + \chi_A) I_k \in M^{k \times k}.$$
The existence of a minimizer is standard. For simplicity of notation, we define
\[ H^1_g(\Omega, N) = \{ n \in H^1(\Omega, \mathbb{R}^k), n(x) \in N \text{ a.e. and } n|_{\partial \Omega} = g \} \]
and we are interested in obtaining some regularity results for the minimizer.

First we derive the Euler-Lagrange equation for a minimizer. Let \( N_c = \{ x \in \mathbb{R}^k, \text{dist}(x, N) < \epsilon \} \) be a small tubular neighborhood of \( N \) on which nearest point projection \( \Pi \) onto \( N \) is well defined. Consider \( n + s\xi \) where \( \xi = (\xi_1, \xi_2, \cdots, \xi_k) \in C^\infty_c(\Omega, \mathbb{R}^k) \). For \( s \) small enough, \( n + s\xi \) lies in \( N_c \) and the following mapping
\[ n^s = \Pi \circ (n + s\xi) \]
is an admissible mapping with
\[ D^n n^s = D^n n + s(d\Pi_n \circ D^n \xi + \text{Hess}\Pi_n(\xi, D^n n)) + o(s). \]

Therefore
\[
0 = \frac{d}{ds}|_{s=0} E(n^s) = \frac{d}{ds}|_{s=0} \int_\Omega a_{\alpha \beta}(x) D^{\alpha} n^s(x) \cdot D^{\beta} n^s(x) dx \\
= 2 \int_\Omega a_{\alpha \beta}(x) D^{\alpha} n(x) \cdot d\Pi_n(D^{\beta} \xi(x)) + a_{\alpha \beta}(x) D^{\alpha} n(x) \cdot \text{Hess}\Pi_n(\xi, D^{\beta} n) \\
= 2 \int_\Omega a_{\alpha \beta}(x) D^{\alpha} n \cdot D^{\beta} \xi - a_{\alpha \beta}(x)(A_n(D^{\alpha} n, D^{\beta} n)) \cdot \xi \\
= 2 \int_\Omega \sum_{\alpha=1}^d \{(1 + \chi_A)D^{\alpha} n \cdot D^{\alpha} \xi - (1 + \chi_A)A_n(D^{\alpha} n, D^{\alpha} n) \cdot \xi\} = 0, \quad (2.2)
\]
here \( A_n \) is the second fundamental form of \( N \) at \( n(x) \).

Partial regularity result for \( n \) then follows from a more general theorem:

**Theorem 2.1.** ([Theorem 1 and 2, [Sh]]) Let \( \Omega \subset \mathbb{R}^d \) be a smooth open set, \( E = \int_\Omega a_{\alpha \beta} D^{\alpha} n \cdot D^{\beta} n, a_{\alpha \beta}(x) \in L^\infty \) satisfying \( \Lambda^{-1} I_d \leq a_{\alpha \beta}(x) \leq \Lambda I_d, \) where \( \Lambda \) is a positive constant, \( I_d \) is the \( d \times d \) unit matrix. Assume \( N \) is a smooth compact Riemannian manifold, \( n \) is an \( E \)-minimizing map from \( \Omega \) to \( N \), then there exists a \( \epsilon = \epsilon(\Lambda) > 0 \) such that if \( \nu^{1/2} \int_{B_r(x)} |\nabla n|^2 \leq \epsilon, \) then \( n \in C^\alpha(B_\frac{r}{2}(x)) \) for some \( 0 < \alpha < 1. \) Thus \( n \) is locally Hölder continuous outside a relatively closed subset \( S_n \) of \( \Omega. \) Moreover, \( H^{d-2}(S_n) = 0. \)

Meyers’ example ([Gi]) show that \( C^\alpha \) regularity for general case is optimal. For our case, the coefficient is piecewise constant, we can actually prove the following lipschitz partial regularity result:

**Theorem 2.2.** Let \( a_{\alpha \beta} = \delta_{\alpha \beta}(c_1 + c_2 \chi_A), c_1 > 0, c_1 + c_2 > 0 \) are given constants. Then any \( E \)-minimizing map \( n \) is locally lipschitz continuous on \( \Omega \backslash S_n. \)

The proof of theorem 2.2 depends on a standard blow up argument and the observation that \( \nabla n \in L^p_{loc} \) for some \( p > 2. \) Our analysis uses strong convergence of the blow up coefficients. We remark that the same arguments therefore is also applicable to the case when \( a_{\alpha \beta} \) is piecewise continuous but would fail in general case when \( a_{\alpha \beta} \) are merely bounded and measurable.

The lipschitz regularity theorem follows from small energy estimates. An important ingredient in proving small energy estimates is the following monotonicity formula. For simplicity of notation, we shall always assume \( a_{\alpha \beta} = \delta_{\alpha \beta}(1 + \chi_A). \)
Denote
\[
E(n, r, x) = \frac{1}{r^{d-2}} \int_{B_r(x)} a_{\alpha\beta}(y) D^\alpha n(y) \cdot D^\beta n(y) dy
\]
\[
= \frac{1}{r^{d-2}} \int_{B_r(x)} (1 + \chi_A) |\nabla n(y)|^2 dy,
\]
we have

**Lemma 2.1.** There are constants \(c\) and \(R_0\) depending only on \(d, A\) such that
\[
E(n, r, x) \leq cE(n, R, x)
\]
for any \(x \in \Omega, \overline{B_R(x)} \subset \Omega\) and all \(r \leq R \leq R_0\).

**Proof of Lemma 2.1:** Note the lemma is trivial for \(d = 2\), we assume \(d > 2\).

**Case I:** \(B_R(x) \subset A\) or \(B_R(x) \subset \Omega \setminus A\). We prove the case when \(B_R(x) \subset \Omega \setminus A\), the other case is proved in the same way. For \(\sigma \in (r, R)\), take comparison map defined by
\[
v_\sigma(x) = \begin{cases} 
\frac{n(x)}{|n|} & |x| < \sigma, \\
n(x) & |x| \geq \sigma.
\end{cases}
\]
By minimality of \(n\), we have
\[
\int_{B_r(x)} |\nabla n|^2 = \int_{B_r(x)} (1 + \chi_A) |\nabla n|^2 \leq \int_{B_r(x)} (1 + \chi_A) |\nabla v|^2 = \int_{B_r(x)} |\nabla v|^2
\]
\[
= (d-2)^{-1} \sigma \left( \int_{\partial B_r(x)} |\nabla n|^2 - \int_{\partial B_r(x)} \frac{|\partial n|}{|r|} |n| \right),
\]
which is
\[
0 \leq \sigma^{2-d} \int_{|x| = \sigma} \left| \frac{\partial n}{\partial r} \right|^2 \leq \frac{d}{d-2} \left( \sigma^{2-d} \int_{B_r(x)} |\nabla n|^2 dy \right), \quad \forall \sigma \in (r, R).
\]
Integrate (2.6) from \(r\) to \(R\), we have
\[
r^{2-d} \int_{B_r(x)} |\nabla n|^2 dy \leq R^{2-d} \int_{B_R(x)} |\nabla n|^2 dy.
\]

**Case II:** \(x \in \partial A\), there exists a \(R_0\) depending only \(A\) such that \(\partial A \cap B(x, R_0)\) can be expressed as a graph of a \(C^\infty\) function for any \(x \in \partial A\). Moreover, \(R_0\) can be chosen in such a way that there exists a \(\lambda > 0, \lambda R_0 \leq \frac{1}{2}\), for any \(w \in H^1(B(x, R_0), \mathbb{R}^k)\) and any \(\sigma \leq R_0\),
\[
(1 - \lambda \sigma) \int_{B_r(x)} (1 + \chi_{B_r^+}) |\nabla w|^2 \leq \int_{B_r(x)} (1 + \chi_A) |\nabla w|^2 \leq (1 + \lambda \sigma) \int_{B_r(x)} (1 + \chi_{B_r^+}) |\nabla w|^2.
\]
For \(R \leq R_0\) and any \(\sigma \in (r, R)\), let
\[
v_\sigma(x) = \begin{cases} 
\frac{n(x)}{|n|} & |x| \leq \sigma, \\
n(x) & |x| > \sigma.
\end{cases}
\]
By minimality of $n$ and (2.7) we have
\begin{align*}
(1 - \lambda \sigma) \int_{B_r(x)} (1 + \chi_{\mathbb{R}^d_+})|\nabla n|^2 &\leq \int_{B_r(x)} (1 + \chi_A)|\nabla n|^2 \\
\leq \int_{B_r(x)} (1 + \chi_A)|\nabla v_\sigma|^2 &\leq (1 + \lambda \sigma) \int_{B_r(x)} (1 + \chi_{\mathbb{R}^d_+})|\nabla v_\sigma|^2 \\
= (1 + \lambda \sigma)(d - 2)^{-1} \sigma \left[ \int_{\partial B_r(x)} (1 + \chi_{\mathbb{R}^d_+})|\nabla n|^2 - \int_{\partial B_r(x)} (1 + \chi_{\mathbb{R}^d_+}) |\partial n|^{2} \right].
\end{align*}

This implies
\begin{align*}
\frac{d}{ds} \left\{ \sigma^{2-d}(1 + \lambda \sigma)^{2(d-2)} \int_{B_r(x)} (1 + \chi_{\mathbb{R}^d_+})|\nabla n|^2 \right\} \\
\geq \sigma^{2-d}(1 + \lambda \sigma)^{2(d-2)} \int_{\partial B_r(x)} (1 + \chi_{\mathbb{R}^d_+}) |\partial n|^2 \geq 0. \quad (2.9)
\end{align*}

Integrate (2.9) from $r$ to $R$, we obtain
\begin{align*}
r^{2-d}(1 + \lambda r)^{2(d-2)} \int_{B_r(x)} (1 + \chi_{\mathbb{R}^d_+})|\nabla n|^2 &\leq R^{2-d}(1 + \lambda R)^{2(d-2)} \int_{B_R(x)} (1 + \chi_{\mathbb{R}^d_+})|\nabla n|^2. \quad (2.10)
\end{align*}

(2.10) together with (2.7) gives
\begin{align*}
r^{2-d}(1 + \lambda r)^{2(d-2)-1} \int_{B_r(x)} (1 + \chi_A)|\nabla n|^2 \\
\leq R^{2-d}(1 + \lambda R)^{2(d-2)-1} \int_{B_R(x)} (1 + \chi_A)|\nabla n|^2. \quad (2.11)
\end{align*}

Inequality (2.4) then follows from (2.11) with $c = 2^d$ and $R_0$ small enough depending only on $A$.

Case III: $x \notin \partial A$ and $|B_R(x) \cap A| > 0, |B_R(x) \cap A^c| > 0$. $R \leq R_0$, here $R_0$ is as in case II.

1) $d(x, \partial A) \geq \frac{1}{4} R$
   a) If $r > \frac{1}{4} R$, then
   \[ \mathbb{E}(n, r, x) \leq 4^{d-2} \mathbb{E}(n, R, x). \]
   b) If $r \leq \frac{1}{4} R < d(x, \partial A)$, we can apply case I to $B_r(x) \subset B_{\frac{1}{4} R}(x)$ and obtain
   \[ \mathbb{E}(n, r, x) \leq \mathbb{E}(n, \frac{1}{4} R, x) \leq 4^{d-2} \mathbb{E}(n, R, x). \]

2) $d(x, \partial A) < \frac{1}{4} R$
   a) If $r \geq \frac{1}{4} R$, we still have
   \[ \mathbb{E}(n, r, x) \leq 4^{d-2} \mathbb{E}(n, R, x). \]
   b) If $d(x, \partial A) \leq r \leq \frac{1}{4} R$, then we can find $y \in \partial A$ such that $B_r(x) \subset B_{2r}(y) \subset B_{\frac{1}{2} R}(y) \subset B_R(x)$. Hence
   \[ \mathbb{E}(n, r, x) \leq 4^{d-2} \mathbb{E}(n, 2r, y) \leq 2^{d-2} \mathbb{E}(n, \frac{R}{2}, y) \leq 4^{d-2} \mathbb{E}(n, R, x). \]
c) If \( r \leq d(x, \partial A) = l < \frac{1}{4}R \), then we can find \( y \in \partial A \) such that 
\[ B_l(x) \subset B_l(x) \subset B_2(y) \subset B_R(x) \], apply case I and case II we have 
\[ E(n, r, x) \leq cE(n, R, x) \].

The lemma then holds for all \( r \leq R \leq R_0 \) where \( c, R_0 \) depends only on \( d, A \). □

**Remark 2.1.** For \( r \leq 1, x_0 \in \mathbb{R}^d \), let \( A_{x_0, r} = \{ x, x_0 + rx \in A \} \). Examine the proof of lemma (2.1) carefully, we see that the same proof shows (2.4) holds for all \( n_r \) with constants \( c, R_0 \) independent of \( r \leq 1, x_0 \), here \( n_r \) is a minimizer of 
\[ \int_{B_1(0)} a(x) \| \nabla n_{x_0, \lambda}(x) \| dx = \frac{1}{\lambda^{d-2}} \int_{B_1(0)} a(x) \| \nabla n(x) \| dx. \]

**Lemma 2.2.** There is a sequence \( \lambda_i \to 0 \), \( \lambda_i \in (0, 1] \) such that \( n_{x_0, \lambda_i} \) converges weakly in \( W^{1,2}(B_1(0), \mathbb{R}^d) \) to a limiting map \( n_{x_0} \in W^{1,2}(B_1(0), \mathbb{R}^d) \) satisfying 
\[ \frac{\partial n_{x_0}}{\partial r} = 0 \text{ a.e. in } B_1(0). \]

**Proof:** The proof follows directly from the monotonicity formula and a similar argument as in [SU1].

From lemma 2.1 can also prove the following Cacciopoli type inequality.

**Lemma 2.3.** Let \( \lambda \) be an energy minimizer of (2.1), \( \Lambda \) be a given constant, if 
\[ R^{d-2} \int_{B_R(x_0)} |\nabla n|^2 \leq \Lambda \]
for some ball \( B_R(x_0) \) with closure contained in \( \Omega \), then 
\[ \rho^{d-2} \int_{B_2(y)} |\nabla n|^2 \leq C \rho^{-d} \int_{B_2(y)} |n - \tau|^2 \]
for each \( y \in B_2 \), \( \rho < \frac{R}{4} \). Here \( C = C(d, N, \Lambda, A) > 0 \).

**Proof:** The proof of lemma 1 in section 2.8 of [Si] can be carried through in our case with only slight changes. We refer the reader to their proof.

A direct result of the Cacciopoli’s inequality is the following reverse Hölder inequality. The proof is standard (see e.g. [Gi]).

**Lemma 2.4.** Let \( a(x) = 1 + \chi_A \). If \( n \) is a minimizing of \( I = \int_\Omega a(x)|\nabla n|^2 dx \) in \( H^1_0(\Omega, \mathbb{R}) \), \( \Omega \) and for some given \( \Lambda \), we have 
\[ R^{2-d} \int_{B_R(x_0)} a(y)|\nabla n|^2 \leq \Lambda, \]
then there exists \( p > 2 \) such that \( |Dn| \in L^p_{\text{loc}}(\Omega) \) and for \( \rho < \frac{R}{4}, y \in B_2(x) \) we have 
\[ \left\{ \int_{B_2(y)} |Dn|^p \right\}^{\frac{1}{p}} \leq C \left\{ \int_{B_2(y)} |Dn|^2 dx \right\}^{\frac{1}{2}}, \]
where \( C, p \) depend only on \( d, N, A, \Lambda \).

To show that \( n \) is locally Lipschitz continuous on \( \Omega \setminus \mathbb{S}_n \). After a suitable translation, rotation and scaling, it reduces to showing the following statement in the normalized situation:
Let \( A = \{(x, y) \in B_{1}^{d-1}(0) \times \mathbb{R} : y > \phi(x)\} \) and \( \phi \) is a \( C^{1, \gamma} \) function on \( B_{1}^{d-1}(0) \) with \( \phi(0) = |\nabla \phi(0)| = 0 \) and \( ||\phi||_{C^{1, \gamma}} \leq 1 \), then any minimizer \( n \) of
\[
\int_{B_{1}(0)}(1 + \chi_{A})|\nabla n|^{2} dx, \quad u \text{ is Lipschitz continuous in } B_{2}(0) \subset \mathbb{R}^{d}.
\]

We let \( ||\phi||_{C^{1, \gamma}(B_{1})} = K(1) \) and define \( K(r) = ||\phi^{r}||_{C^{1, \gamma}(B_{1})}, \) for \( 0 < r < 1 \), where \( \phi^{r}(x) = \frac{1}{r}\phi(rx) \). Thus \( K(r) \leq r^{1}K(1), 0 < r < 1 \). We then have the following statement.

**Lemma 2.5.** Let \( a(x) = 1 + \chi_{A}, \lambda \leq 1 \). There exists constant \( \delta_{0}, \theta \in (0, 1), \mu \in (0, 1) \) depending only on \( d, N \) such that for any minimizer \( n_{\lambda} \) of
\[
I_{\lambda} = \int_{B_{1}(0)} a(\lambda x)|\nabla n|^{2} dx \text{ satisfying }
\]
\[
\int_{B_{1}(0)} a(\lambda y)|\nabla n_{\lambda}(y)|^{2} dy \leq \delta_{0},
\]
we have
\[
\frac{1}{\theta^{2d-2}} \int_{B_{1}(0)} a(\lambda y)|\nabla n_{\lambda}|^{2} \leq \int_{B_{1}(0)} a(\lambda x)|\nabla n_{\lambda}|^{2},
\]
and
\[
\int_{B_{2}(0)} |D^{\lambda}n_{\lambda} - D^{\lambda}n_{\lambda,B}(0)|^{2} \leq \theta^{2\mu} \int_{B_{1}(0)} |D^{\lambda}n_{\lambda} - D^{\lambda}n_{\lambda,B}(0)|^{2},
\]
here \( D^{\lambda}n_{\lambda} = \{1 + \chi_{A_{\lambda}}\}D_{\lambda}m_{\lambda}, D_{1}n_{\lambda}, \cdots, D_{d-1}n_{\lambda} \}, D_{i}n = \frac{\partial n}{\partial x_{i}}, i = 1, \cdots, d. \)
\( A_{\lambda} = \{x, \lambda x \in A\}. \)

**Proof:** (2.12) follows from small energy estimates in [Sh] (Proof of theorem 1 in [Sh]). We prove (2.13) by a blow up argument. If (2.13) were not true, there would exist \( \epsilon_{k}, n_{k}, \lambda_{k} \) such that \( n_{k} \) is a minimizer of \( \int_{\Omega}(1 + \chi_{A_{\lambda_{k}}})|\nabla n|^{2} \) with
\[
\int_{B_{1}(0)} (1 + \chi_{A_{\lambda_{k}}})|\nabla n_{k}|^{2} = \epsilon_{k} \leq 0 \quad (2.14)
\]
but
\[
\int_{B_{2}(0)} |D^{\lambda_{k}}n_{k} - D^{\lambda_{k}}n_{k,B}(0)|^{2} > \theta^{2\mu} \int_{B_{1}(0)} |D^{\lambda_{k}}n_{k} - D^{\lambda_{k}}n_{k,B}(0)|^{2}.
\]
Let
\[
m^{k}(x) = \frac{n_{k}(x) - a_{k}}{\epsilon_{k}}, \quad a_{k} = \int_{B_{1}(0)} n_{k} dx. \quad (2.15)
\]
Then \( m^{k} \) is a bounded sequence in \( H^{1}(B_{1}(0), \mathbb{R}^{d}) \). Passing to a subsequence if necessary, we may assume \( m^{k} \) converges weakly to \( m \in H^{1}(B(0,1), \mathbb{R}^{d}) \). Since each \( A_{\lambda} \) is a scaling of \( A \) with a scaling constant smaller than one and \( A \) is a smooth set, the perimeter \( P(A_{\lambda}, B_{1}(0)) \) is finite and we can assume \( \chi_{A_{\lambda_{k}} \cap B_{1}(0)} \rightharpoonup \chi_{A_{\lambda_{k}}} \rightharpoonup B_{1}(0) \). Since \( n_{k} \) is a minimizer of \( \int_{\Omega}(1 + \chi_{A_{\lambda_{k}}})|\nabla n|^{2} \), we have
\[
\int_{B_{1}(0)} \sum_{\alpha=1}^{d} (1 + \chi_{A_{\lambda_{k}}})D^{\alpha}m^{k} \cdot D^{\alpha} \eta dx = \epsilon_{k} \int_{B_{1}(0)} \sum_{\alpha=1}^{d} (1 + \chi_{A_{\lambda_{k}}})A_{n_{k}}(D^{\alpha}m^{k}, D^{\alpha}m^{k}) \eta dx \quad (2.16)
\]
and
\[
\int_{B_{1}(0)} \sum_{\alpha=1}^{d} (1 + \chi_{A_{\lambda_{k}}})D^{\alpha}m \cdot D^{\alpha} \eta dx = 0 \quad (2.17)
\]
for any $\eta \in C^\infty_0(B(0,1),\mathbb{R}^k)$. Subtracting (2.17) from (2.16) we find
\[\int_{B_1} \sum_{\alpha=1}^d \left\{ (1 + \chi_{A_k}) D^\alpha m^k \cdot D^\alpha \eta - (1 + \chi_{B_1}) D^\alpha m \cdot D^\alpha \eta \right\} dx = \epsilon_k \int_{B_1} \sum_{\alpha=1}^d (1 + \chi_{A_k}) A_{n_k}(D^\alpha m^k, D^\alpha m^k) \eta dx. \tag{2.18}\]

By (2.14) and lemma 2.4, we can find some $p > 2$ depending only on $d, N$,
\[\left(\int_{B(0, \frac{1}{4})} |\nabla n_k|^p dx\right)^{\frac{1}{p}} \leq C \left(\int_{B(0,1)} |\nabla n_k|^2 dx\right)^{\frac{1}{2}} \tag{2.19}\]
for some constant $C$ depending only on $d, N$.

After rescaling, (2.19) reads
\[\left(\int_{B(0, \frac{1}{2})} |\nabla m^k|^p dz\right)^{\frac{1}{p}} \leq C \left(\int_{B(0,1)} |\nabla m^k|^2 dz\right)^{\frac{1}{2}}. \tag{2.20}\]

It follows that $|\nabla m^k|$ is bounded in $L^p(B_{\frac{1}{2}}(0))$. Moreover, a similar argument as in lemma 4.1 of [Ev1], we conclude that $m^k$ is bounded in $L^s(B_1(0))$ for all $1 \leq s < \infty$. Let $q$ satisfy $\frac{2}{q} + \frac{1}{q} = 1$. By approximation the identity (2.18) holds for $\eta \in H^1_0(B_1(0), \mathbb{R}^k) \cap L^q(B_1(0), \mathbb{R}^k)$. We now insert $\eta = \xi^2(m^k - m)$ into (2.18). Here $\xi \equiv 1$ in $B_{\frac{3}{4}}(0)$ and $\xi \equiv 0$ outside $B_1(0)$.

The left hand side of (2.18) is
\[L_k \geq \epsilon_k \int_{B(0, \frac{1}{2})} |\nabla m^k - \nabla m|^2 dx + o(1). \tag{2.21}\]

The last inequality follows from the fact that $m^k \to m$ strongly in $L^2(B_1(0))$, $\nabla m^k, \nabla m$ are bounded in $L^p(B(0, \frac{1}{2}))$ and $\chi_{A_k} \to \chi_{B_1^*}$ strongly in $L^q(B_1(0))$.

The right hand side of (2.18) reads
\[R_k = \epsilon_k \int_{B_1} \sum_{B_i(0)} (1 + \chi_{A_k}) A_{n_k}(D^\alpha m^k, D^\alpha m^k) \xi^2(m^k - m) dx \leq \epsilon_k C \int_{B_1} |\nabla m^k|^2 \xi^2 |m^k - m|, \tag{2.22}\]

Combine (2.21) and (2.22) we obtain
\[\nabla m^k \to \nabla m \text{ strongly in } L^2(B_{\frac{1}{4}}(0)). \tag{2.23}\]
Since $m$ is a weak solution of (2.17), we can find some $\alpha \in (0, 1)$ such that
\[
\int_{B_{\frac{1}{2}}(0)} |D^2 m - \overline{D^2 m}|^2 \leq C\theta^{2\alpha} \int_{B_{\frac{1}{4}}(0)} |D^2 m - \overline{D^2 m}|^2,
\tag{2.24}
\]
where $D^2 m = (1 + |\chi_{R^2}|) D_\mu m, D_1 m, \ldots, D_{d-1} m)$. Pick $\mu < \alpha$, choose $\theta$ sufficiently small, a contradiction will then arise from the strong convergence of $\nabla m^k$ to $\nabla m$ in $L^2$ and strong convergence of $\chi_{A_k}$ to $\chi_{R^2}$ in $L^2$. $\square$

A standard iteration argument then gives Lipschitz regularity for $n$. Moreover, it gives the following estimates on the gradients.

**Lemma 2.6.** Let $n$ be a minimizer of (2.1). There exists $\delta > 0$ such that if
\[
\frac{1}{\epsilon^3} \int_{B_{\epsilon}(x)} |\nabla n|^2 \geq \delta, \quad \text{then} \quad n \text{ is Lipschitz continuous in } B_{\frac{1}{2}}(x) \text{ with}
\]
\[
|\nabla n|_{L^\infty(B_{\frac{1}{2}}(x))} \leq C \left( \frac{1}{\epsilon^3} \int_{B_{\epsilon}(x)} |\nabla n|^2 \right)^{\frac{1}{2}}. \quad \text{Here } C = C(d, N).
\]

Further more, we could reduce the dimension for the singular set of $n$. First we quote the following lemmas from Simon’s lecture notes [Si], which is originally due to Luckhause ([Lu1, Lu2]).

**Lemma 2.7** (Corollary 1, [Si], page 27). Let $N$ be a smooth compact manifold embedded in $\mathbb{R}^p$ and $\Lambda > 0$. There are $\delta_0 = \delta_0(n, N, \Lambda)$ and $C = C(n, N, \Lambda)$ such that the following hold:

1. If we have $\epsilon \in (0, 1)$ and if $u \in W^{1,2}(B_\rho(y); N)$ with $\rho^{2-n} \int_{B_\rho(y)} |\nabla u|^2 \leq \Lambda$, and $\epsilon^{2-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 \leq \delta_0^2$, then there is $\sigma \in (\frac{3}{4}, \rho)$ such that there is a function $w = w_\sigma \in W^{1,2}(B_\rho(y); N)$ which agrees with $u$ in a neighborhood of $\partial B_\sigma(y)$ and which satisfies
\[
\sigma^2 \int_{B_{\sigma}(y)} |Dw|^2 \leq \epsilon \rho^{2-n} \int_{B_\rho(y)} |Du|^2 + \epsilon^{-1} C \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2.
\]

2. If $\epsilon \in (0, \delta_0)$, and if $u, v \in W^{1,2}(B_{(1+\epsilon)}\rho(y); B_\rho(y); N)$ satisfy the inequalities
\[
\rho^{2-n} \int_{B_{(1+\epsilon)}\rho(y)\setminus B_\rho(y)} |Dw|^2 \leq \Lambda \quad \text{and}
\]
\[
\epsilon^{2-n} \rho^{-n} \int_{B_{(1+\epsilon)}\rho(y)\setminus B_\rho(y)} |u-v|^2 \leq \delta_0^2,
\]
then there is $w \in W^{1,2}(B_{(1+\epsilon)}\rho(y)\setminus B_{\rho(y)}; N)$ such that $w = u$ in a neighborhood of $\partial B_{\rho}(y)$, $w = v$ in a neighborhood of $\partial B_{(1+\epsilon)\rho}(y)$, and
\[
\rho^{2-n} \int_{B_{(1+\epsilon)\rho}(y)\setminus B_{\rho}(y)} |Dw|^2 \leq C \rho^{2-n} \int_{B_{(1+\epsilon)\rho}(y)\setminus B_{\rho}(y)} (|Du|^2 + |Dv|^2) + Ce^{-2} \rho^{-n} \int_{B_{(1+\epsilon)\rho}(y)\setminus B_{\rho}(y)} |u-v|^2.
\]

**Lemma 2.8.** There exists a sequence $\lambda_i \to 0$ such that the maps $n_{a,\lambda_i}$ defined by
\[
n_{a,\lambda_i}(x) = n(a + \lambda_i x) \quad \text{for } x \in B_1(0)
\]
converges strongly in $H^1(B_1(0), N)$ to a map $n_a \in H^1(B_1(0), N)$ which is homogeneous of degree 0. Moreover, if $\text{dist}(a, \partial A) > 0$, then $n_a$ is a minimizing harmonic map; if $a \in \partial A$, then $n_a$ is a minimizing map of $\int_{B_1(0)} (1 + |\chi_{R^2}|) |\nabla n|^2$.

**Proof:** The argument in section 2.9 of [Si] can be carried over with only slight modification. We refer the reader to their proof.
Theorem 2.3. Let \( a(x) = (c_1 + c_2 \chi_A) \) with \( c_1 > 0, c_1 + c_2 > 0 \) and \( A \) being a smooth subset of \( \Omega \). Then the interior singular set \( S_n \) for any minimizer \( u \) of \( \int_{\Omega} a(x)|\nabla u|^2 \) has Hausdorff dimension less than or equal to \( d-3 \), in particular, \( S_n \) is a discrete set of points when \( d = 3 \).

Proof: We can follow essentially the same argument of [SU1] section 5 or Theorem 4.5 of [HL]. We refer readers to their papers. □

Under the additional assumption that \( g \in C^{1,\alpha}(\partial \Omega) \), we can have the following.

Theorem 2.4. Let \( g \in C^{1,\alpha}(\partial \Omega, N) \). If \( n \) is a minimizer of \( \int_{\Omega}(1 + \chi_A)|\nabla n|^2 \) in \( H^1_0(\Omega, N) \), then the singular set \( S_n \) of \( n \) is a compact subset of the interior of \( \Omega \); in particular, \( n \) is \( C^{1,\alpha} \) in a full neighborhood of \( \partial \Omega \).

Proof: Note \( A \subset \Omega \), the same argument in [SU2] applies in our case and the boundary regularity of \( n \) follows. □

In general case where \( a_{\alpha\beta}(x) \) is only bounded and measurable, the monotonicity formula is lacking, we can not carry out the above argument to further reduce the dimension of \( S_n \). Nonetheless, under additional assumptions on \( N \), this can be done. Assume \( N \) is a simply connected smooth compact submanifold of \( \mathbb{R}^k \), \( a_{\alpha\beta}(x) \) are bounded measurable functions. We consider the regularity of a minimizer of \( \int_{\Omega} a_{\alpha\beta}(x) D^a n \cdot D^b n \) in \( H^1_0(\Omega, N) \).

First we quote the following extension lemma from [HL] (a simple version in the case \( n = S^2 \) can be found in [HKL])

Lemma 2.9 (Theorem 6.2, [HL]). Let \( N \) be a simply connected smooth compact submanifold of \( \mathbb{R}^k \). If \( u \in W^{1,2}(\Omega, N) \) and \( a \in \Omega \), then for almost every positive \( r < \text{dist}(a, \partial \Omega) \), there is a function \( w \in W^{1,2}(B_r(a), N) \) such that \( w = u \) on \( \partial B_r(a) \) and

\[
\int_{B_r(a)} |Dw|^2 \leq C \left( \int_{\partial B_r(a)} |\nabla_{\text{tan}} u|^2 \cdot \int_{\partial B_r(a)} |u - \xi|^2 dS \right)^{\frac{1}{2}},
\]

where \( \xi \in \mathbb{R}^k \) is arbitrary and \( C \) is an absolute constant.

Lemma 2.10. There exists a positive constant \( C = C(d, N) \) such that for any minimizer \( n \) of \( \int_{\Omega} a(x)|\nabla n|^2 \) in \( H^1_0(B_1(0), N) \), we have the following uniform energy bound:

\[
\int_{B_r(0)} |\nabla n|^2 \leq \frac{C(d, N)}{1-r} \quad \text{for } 0 \leq r < 1.
\]

Proof: The proof of Theorem 3.1 in [HKL] can be carried over directly to our case. □

Let \( E = \int_{B_R} a_{\alpha\beta}(x) D^a v_i D^b v_j dx \), \( \mathcal{F} = \{ \Sigma, \Sigma \subset B_R \text{ closed and } \Sigma \subset \text{singv} \text{ for some minimizer } v \text{ of } E \} \), then the following hold (for a proof, see e.g. [L2]):

Lemma 2.11. •

- a) If \( \Sigma \in \mathcal{F} \), then \( \Sigma - \lambda \subset B_R \in \mathcal{F} \) for \( |x| < R, 0 < \lambda < R - |x| \).
- b) \( \mathcal{F} \) is compact under the Hausdorff metric.
- c) \( H^{n-2}(\Sigma) = 0 \) for all \( \Sigma \in \mathcal{F} \).

Note that a direct result of the lemma is that there exists a \( \delta = \delta(N) > 0 \) such that \( H^{n-2-\delta}(\Sigma) = 0 \) for all \( \Sigma \in \mathcal{F} \).
3. Homogenization case. In this section, we return to the homogenization problem. As in classical theory of homogenization, we are interested in determining the asymptotic behavior of solutions to the above minimization problem. Typically, this analysis amounts to the knowledge of apriori bounds on the norms of the solutions which are valid uniformly in the small parameter $\epsilon$ and ensure the compactness of the family $\{n^\epsilon\}_{\epsilon>0}$ in a suitable function space. Before we prove the main results, we first introduce some notations used in this section. We shall always use Einstein’s summation principle in this section.

$$E_\epsilon(n, r, x) = \frac{1}{r^{n-2}} \int_{B_r(x)} a_{\alpha\beta} \left( \frac{y}{\epsilon} \right) D^{\alpha} n(y) \cdot D^{\beta} n(y) dy,$$

$$I_\epsilon = a'(n, n) = \int_{\Omega} a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^{\alpha} n(x) \cdot D^{\beta} n(x) dx,$$

$$Y : \text{unit cell in } \mathbb{R}^d,$$

$$(f) = \frac{1}{|Y|} \int_Y f(x) dx,$$

$$(u, v) = \int_{\Omega} u \cdot v dx.$$

We define

$$W(Y) = \{ \phi | \phi \in H^1(Y, \mathbb{R}^k), \phi = (\phi_1, \cdots, \phi_k) \text{ periodic in } Y \}$$

for $\phi, \psi \in H^1(Y, \mathbb{R}^k)$, we set

$$a_1(\phi, \psi) = \int_Y a_{\alpha\beta}(y) D^{\alpha} \phi(y) \cdot D^{\beta} \psi(y) dy,$$

and we introduce

$$P^\beta_j(y) = \{ 0, \cdots, 0, y^\beta, 0, \cdots, 0 \}$$

with $\beta \in \mathbb{R}^d, |\beta| = 1, 1 \leq j \leq k$ and define

$$\chi_j^\beta \in W(Y), \quad \text{such that}$$

$$a_1(\chi_j^\beta - P^\beta_j, \psi) = 0 \quad \forall \psi \in W(Y). \quad (3.1)$$

Since $\chi_j^\beta$ is uniquely defined up to a constant, the following quantity is uniquely defined

$$q^{ij}_{\alpha\beta} = \frac{1}{|Y|} a_1(\chi_j^\beta - P^\beta_j, \chi_i^\alpha - P^\alpha_i) \quad (3.2)$$

and

$$a(u, v) = \int_{\Omega} q^{ij}_{\alpha\beta} D^{\alpha} u_i D^{\beta} v_j dx.$$
We then have the formula (See e.g. [BLP])

\[
a^{ij}_{\alpha \beta} = \frac{1}{|V|} a^{ij}_i (\chi^{*} - \chi^{*}_{\alpha} - \chi^{*}_{\beta} + \chi^{*}_{\beta}).
\]

(3.4)

We shall also need some standard results and notations from [BLP]. We denote

\[A^* = -\frac{\partial}{\partial x^\alpha} \left( a^{ij}_{\alpha \beta} \left( \frac{x}{\epsilon} \right) \frac{\partial}{\partial x^\beta} \right).\]

We expand \(A^* = \epsilon^{-2} A_1 + \epsilon^{-1} A_2 + \epsilon^0 A_3\), where

\[
A_1 = -\frac{\partial}{\partial y^\alpha} \left( a^{ij}_{\alpha \beta}(y) \frac{\partial}{\partial y^\beta} \right),
\]

\[
A_2 = -\frac{\partial}{\partial y^\alpha} \left( a^{ij}_{\alpha \beta}(y) \frac{\partial}{\partial y^\beta} \right) - \frac{\partial}{\partial x^\alpha} \left( a^{ij}_{\alpha \beta}(y) \frac{\partial}{\partial y^\beta} \right),
\]

\[
A_3 = -\frac{\partial}{\partial x^\alpha} \left( a^{ij}_{\alpha \beta}(y) \frac{\partial}{\partial x^\beta} \right).
\]

\(A^*\) denotes the adjoint operator of \(A\).

3.1. Homogenization limit. In this section, we prove the following theorem about the homogenization limit.

**Theorem 3.5.** For any sequence \(\{n^\ell\}\), where \(n^\ell\) is a minimizer \(I_\epsilon\), there exists a subsequence \(n^{\epsilon \ell}\) such that \(n^{\epsilon \ell}\) converges weakly to a minimizing harmonic map \(n\) in \(H^2_0(\Omega, N)\). Moreover, there exists some constant \(a_0 > 0\) uniquely determined by \(a^{\alpha \beta}\) such that

\[
\lim_{\epsilon \to 0} \int_{\Omega} a_{\alpha \beta} \left( \frac{x}{\epsilon} \right) D^n a^{\alpha \beta} \cdot D^n n^\ell dx \to a_0 \int_\Omega |\nabla n|^2.
\]

We shall prove the theorem in two steps. First we show that \(n\) is a weakly harmonic map (lemma 3.12), we then show that \(n\) is a minimizing harmonic map and the energy convergence results (lemma 3.15).

**Lemma 3.12.** For any sequence of \(\{n^\ell\}\), \(n^\ell\) being a minimizer of \(I_\epsilon\) in \(H^1_0(\Omega, N)\), there exists a subsequence \(n^{\epsilon \ell}\) such that \(n^{\epsilon \ell}\) converges weakly in \(H^1_0(\Omega, N)\) to a weakly harmonic map \(n\).

**Proof:** Let \(\ell\) be a subsequence such that

\[
\int_{\Omega} a_{\alpha \beta} \left( \frac{x}{\epsilon \ell} \right) D^n a^{\alpha \beta} \cdot D^n n^\ell \to \liminf_{\epsilon \to 0} \int_{\Omega} a_{\alpha \beta} \left( \frac{x}{\epsilon} \right) D^n a^{\alpha \beta} \cdot D^n n^\ell.
\]

By assumption we have

\[
\int_{\Omega} |\nabla n^\ell|^2 \leq \int_{\Omega} a_{\alpha \beta} \left( \frac{x}{\epsilon \ell} \right) D^n a^{\alpha \beta} \cdot D^n n^\ell dx \leq 2 \int_{\Omega} |\nabla n|^2 dx \leq C.
\]

Therefore \(n^\ell\) is a bounded sequence in \(H^2_0(\Omega, \mathbb{R}^k)\), hence a subsequence (we still denote by \(n^\ell\) ) converges weakly in \(L^2(\Omega, \mathbb{R}^k)\), strongly in \(L^2(\Omega, \mathbb{R}^k)\) and pointwise almost everywhere to \(n \in H^2_0(\Omega, \mathbb{R}^k)\). Since \(n^\ell\) is a minimizer of \(\int_{\Omega} a_{\alpha \beta} \left( \frac{x}{\epsilon} \right) D^n a^{\alpha \beta} \cdot D^n n dx \) in \(H^2_0(\Omega, N)\), \(n^\ell\) is a weak solution of the following Euler-Lagrange equation:

\[
-D^{\beta} \left( a_{\alpha \beta} \left( \frac{x}{\epsilon \ell} \right) D^n a^{\alpha \beta} \right) = a_{\alpha \beta} \left( \frac{x}{\epsilon \ell} \right) (A_{n^\ell} (D^n a^{\alpha \beta}, D^n n^\ell)).
\]
To illustrate the main idea, from now on, we assume \( N = S^n \), the general target case could be proved similarly (though technically more complicated). In this case, \( n^i \) is a weak solution of

\[
- \text{div}(c_1 + c_2 \chi_{\Omega_i}) \nabla n^i_j = (c_1 + c_2 \chi_{\Omega_i})|\nabla n^i_j|^2 n^i_j, \quad 1 \leq j \leq k. \tag{3.5}
\]

Here \( \Omega = \{ x, x \in \Omega_0 \} \).

Following the idea of [Ev1], we write equation (3.5) in the form

\[
- \text{div}((c_1 + c_2 \chi_{\Omega_i}) \nabla n^i_j) = (c_1 + c_2 \chi_{\Omega_i})|\nabla n^i_j|^2 n^i_j = \sum_{\alpha=1}^d \sum_{q=1}^k (c_1 + c_2 \chi_{\Omega_i}) \left( \frac{\partial n^i_j}{\partial x^\alpha} \frac{\partial n^i_j}{\partial x^\alpha} n^i_j - \frac{\partial n^i_j}{\partial x^\alpha} n^i_j \right).
\]

Let

\[
b^i_{\alpha,\beta} = (c_1 + c_2 \chi_{\Omega_i}) \left\{ \frac{\partial n^i_j}{\partial x^\alpha} n^i_j - \frac{\partial n^i_j}{\partial x^\alpha} n^i_j \right\}, \tag{3.6}
\]

then from the following lemma 3.13, one concludes that \( b^i_{\alpha,\beta} \) satisfies

\[
\text{div}(b^i_{\alpha,\beta}) = 0 \text{ weakly for each } 1 \leq q, j \leq k.
\]

Denote \( a_{\alpha\beta}(x) = (a^i_{\alpha\beta}(x)) \), set

\[
\xi_{\beta} = a^i_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha n^i,
\]

we see \( \xi_{\beta} \) is bounded in \( L^2(\Omega) \). Therefore we can extract a subsequence, we still denote by \( \xi_{\beta} \) for simplicity of notation, such that

\[
\xi_{\beta} \rightarrow \xi_{\beta} \text{ weakly in } L^2(\Omega).
\]

Taking into account that \( n^i \) is bounded in \( L^\infty \) and converge strongly in \( L^2 \) to \( n \), we obtain

\[
b^i_{\alpha,\beta} \rightarrow \xi_{\alpha} n_j - \xi_{\alpha} n_q \text{ weakly in } L^2.
\]

For each \( q, j \), apply the Div-Curl lemma (see e.g. [Ev2] or [Mu]) to \( \sum_{\alpha=1}^d \frac{\partial n^i_j}{\partial x^\alpha} b^i_{\alpha,\beta} \) we obtain

\[
\frac{\partial n^i_j}{\partial x^\alpha} b^i_{\alpha,\beta} \rightarrow \frac{\partial n^i_j}{\partial x^\alpha} (\xi_{\alpha} n_j - \xi_{\alpha} n_q) \text{ in } \mathcal{D}'(\Omega).
\]

Therefore the limit equation for \( \xi_{\beta} \) is

\[
\int_{\Omega} \xi_{\beta} \frac{\partial \phi}{\partial x^\alpha} dx = \sum_{\alpha=1}^d \sum_{q=1}^k \int_{\Omega} \frac{\partial n^i_j}{\partial x^\alpha} (\xi_{\alpha} n_j - \xi_{\alpha} n_q) \phi dx, \quad \forall \phi \in C_0^\infty(\Omega). \tag{3.7}
\]

We compute \( \xi_{\beta} \) using adjoint functions. We introduce

\[
P = \{ P_j(y) \}_{j=1}^k, \quad P_j(y) = \text{homogeneous polynomial of degree 1},
\]

and we define \( w \) such that

\[
A^*_1 w = 0 \text{ in } Y, \quad w - P \in W(Y). \tag{3.8}
\]

If we set

\[
w - P = - \chi \tag{3.9}
\]

then the equation (3.8) is equivalent to

\[
a^*_1(\chi - P, \psi) = 0 \quad \forall \psi \in W(Y).
\]
We then introduce
\[ w_\varepsilon(x) = \{ \varepsilon w_j \left( \frac{x}{\varepsilon} \right) \}. \]

We observe that
\[ A^\varepsilon w_\varepsilon = 0, \quad (3.10) \]
and that
\[ w_\varepsilon(x) = P(x) - \{ \varepsilon \chi_j \left( \frac{x}{\varepsilon} \right) \}. \]

For \( \phi \in C_0^\infty(\Omega) \), we set
\[ \phi n = \{ \phi n_1, \ldots, \phi n_k \}. \]

Choose
\[ v = \phi w_\varepsilon \]
as a test function in (3.7), and multiply (3.10) by \( \phi n^\varepsilon \), we obtain
\[ \int_\Omega \xi_j^\varepsilon D^\varepsilon (\varepsilon w_j) - \phi D^\varepsilon w_j \, dx - \int_\Omega a_{ij}^\varepsilon \left( \frac{x}{\varepsilon} \right) D^\varepsilon w_j (\phi \varepsilon^\xi_j - \phi D^\alpha \varepsilon^\xi_j) \, dx = \int_\Omega D^\alpha n_j \phi w_\varepsilon \, dx. \quad (3.11) \]

But one verifies that
\[ D^\varepsilon (\varepsilon w_j) - \phi D^\varepsilon w_j \to D^\varepsilon (\phi P_j) - \phi D^\varepsilon P_j \quad \text{strongly in } L^2(\Omega), \]
\[ D^\varepsilon (\phi \varepsilon^\xi_j) - \phi D^\varepsilon n_j \to D^\varepsilon (\phi n_j) - \phi D^\varepsilon n_j \quad \text{strongly in } L^2. \]

and that
\[ \int_\Omega b_{ij}^\varepsilon \phi w_\varepsilon \, dx = \int_\Omega D^\alpha n_j \phi \chi_j \left( \frac{x}{\varepsilon} \right) \, dx - \int_\Omega D^\alpha n_j \varepsilon^\xi_j - \varepsilon^\xi_j n_q \phi P_j. \quad (3.12) \]

The last part of (3.12) follows from the fact that
\[ D^\varepsilon n_j \varepsilon^\xi_j \to D^\varepsilon n_j (\xi_j^\varepsilon n_j - \xi_j^\varepsilon n_q) \quad \text{in } \mathcal{D}'(\Omega) \]
and that
\[ \int_\Omega |D^\varepsilon n_j b_{ij}^\varepsilon \phi \chi_j \left( \frac{x}{\varepsilon} \right)| \, dx \leq C(N) \| \chi \|_{L^\infty} \| \nabla n^\varepsilon \|_{L^2}. \]

On the other hand, as \( \varepsilon \to 0 \),
\[ a_{ij}^\varepsilon \left( \frac{x}{\varepsilon} \right) D^\varepsilon w_j \left( \frac{x}{\varepsilon} \right) = \left( a_{ij}^\varepsilon D^\varepsilon \chi_j \right) \left( \frac{x}{\varepsilon} \right) \to (a_{ij}^\varepsilon D^\varepsilon w_j) \]
in \( L^\infty \) weak star, so that passing to the limit in (3.11) gives
\[ \int_\Omega \xi_j^\varepsilon (D^\varepsilon (\phi P_j) - \phi D^\varepsilon P_j) \, dx - (a_{ij}^\varepsilon D^\varepsilon w_j) \int_\Omega (D^\varepsilon (\phi n_j) - \phi D^\varepsilon n_j) \, dx = \int_\Omega D^\alpha n_j (\xi_j^\varepsilon n_j - \xi_j^\varepsilon n_q) \phi P_j \, dx. \quad (3.13) \]

But \( \int_\Omega D^\varepsilon (\phi n_j) \, dx = 0 \) and the right hand side of (3.13) equals \( \int_\Omega \xi_j^\varepsilon D^\varepsilon (\phi P_j) \, dx \), therefore (3.13) reduces to
\[ - \int_\Omega \xi_j^\varepsilon D^\varepsilon P_j \phi + (a_{ij}^\varepsilon D^\varepsilon w_j) \int_\Omega \phi D^\varepsilon n_j \, dx = 0, \]

We observe that
\[ D^\varepsilon w_\varepsilon \to D^\varepsilon P \quad \text{strongly in } L^2(\Omega). \]
i.e.
\[ \xi^j_\beta D^3 \beta_j = (a_{\alpha \gamma}^j \gamma_w^\alpha w_k) D^\alpha n_i. \] (3.14)

We now take \( P = P_j^\beta \), then \( w = P_j^\beta - \chi_j^\beta \) and (3.14) gives
\[
\xi^j_\beta = \frac{1}{|Y|} a^1_1 (\chi_j^\beta - P_j^\beta - P_i^\alpha) D^\alpha n_i,
\]
so that (using (3.2), (3.3) and (3.4))
\[
(\xi^j_\beta, D^\beta v_j) = a(n, v) \quad \forall v \in W^{1,2}_0(\Omega, \mathbb{R}^k),
\]
and \( n \) is therefore a weak solution of
\[-\text{div}(a_0 \nabla n) = a_0 |\nabla n|^2 n.\]

**Remark 3.3.** For general compact manifold \( N \), we can basically follow the same idea used above to show that the weak limit \( n \) is a weakly harmonic map. But we have to adapt to the work of [Be] to choose appropriate orthonormal frame on \( T_n(x) \) to rewrite the equation (3.5) into a similar form as (3.8). We then can prove the homogenization limit \( n \) is a weakly harmonic map.

**Lemma 3.13.** For each \( \phi \in C_0^\infty(\Omega) \), \( b_{ij}^{\alpha \gamma} \) defined by (3.6), we have
\[
\int_{\Omega} b_{ij}^{\alpha \gamma} D^\alpha \phi(x) dx = 0
\]
for all \( 1 \leq q, j \leq k \).

**Proof:** We compute
\[
\int_{\Omega} b_{ij}^{\alpha \gamma} D^\alpha \phi(x) dx = \sum_{\alpha = 1}^d \int_{\Omega} \frac{\partial \phi}{\partial x_\alpha} \left( c_1 + c_2 \chi_{\alpha}, n^\gamma_j \right) \left( \frac{\partial n^\gamma_j}{\partial x_\alpha} n^\gamma_j - \frac{\partial n^\gamma_j}{\partial x_\alpha} n^\gamma_j \right) dx
\]
\[
= \sum_{\alpha = 1}^d \left( c_1 + c_2 \chi_{\alpha}, \frac{\partial n^\gamma_j}{\partial x_\alpha} \right) \frac{\partial n^\gamma_j}{\partial x_\alpha} - \int_{\Omega} (c_1 + c_2 \chi_{\alpha}) \frac{\partial n^\gamma_j}{\partial x_\alpha} \frac{\partial (\phi n^\gamma_j)}{\partial x_\alpha}
\]
\[
= \int_{\Omega} (c_1 + c_2 \chi_{\alpha}) |\nabla n^\gamma_j|^2 n^\gamma_j \phi - \int_{\Omega} (c_1 + c_2 \chi_{\alpha}) |\nabla n^\gamma_j|^2 n^\gamma_j \phi dx
\]
\[= 0.\]

**Lemma 3.14.** Let \( \chi^\alpha = \{ \chi^\alpha_k \} \) be given by (3.1). If \( n^\gamma \) is a minimizer of \( I_\gamma \) and \( n^\gamma \rightarrow n \) weakly in \( W^{1,2}(\Omega, N) \), then \( \forall F(x) = (F^j_\gamma(x)) \in W^{1,2} \cap L^\infty(\Omega, M^{k \times d}) \),
\[
\lim_{\epsilon \to 0} \int_{\Omega} a^{ij}_{\alpha \beta} \left( \frac{x}{\epsilon} \right) D^\alpha \chi_{\gamma l} \left( \frac{x}{\epsilon} \right) F^j_\gamma(x) D^\beta n^\gamma_j(x) dx = 0.
\]

Here \( M^{k \times k} \) being the set of all \( k \times k \) matrices.
Proof: Let $\phi \in C^\infty_0(\Omega)$. Since $n^\epsilon$ is a weak solution of the Euler-Lagrange equation (2.2), we have
\[
\int_\Omega \frac{a_{ij}^\epsilon}{\epsilon} D^\beta n_i^\epsilon(x) D^\beta n_j^\epsilon(x) \phi(x) dx = 0.
\]
By using Lemma 3.12, we have
\[
\int_\Omega \frac{a_{ij}^\epsilon}{\epsilon} D^\beta n_i^\epsilon(x) D^\beta n_j^\epsilon(x) \phi(x) dx 
= \epsilon \int_\Omega \frac{a_{ij}^\epsilon}{\epsilon} \chi_{M} \left( \frac{x}{\epsilon} \right) F_i^\epsilon(x) \phi(x) dx 
- \epsilon \int_\Omega \frac{a_{ij}^\epsilon}{\epsilon} \chi_{M} \left( x - \frac{d(x, \partial \Omega)}{\epsilon} \right) F_i^\epsilon(x) \phi(x) dx
\]
and we have
\[
\int_\Omega \frac{a_{ij}^\epsilon}{\epsilon} D^\beta n_i^\epsilon(x) D^\beta n_j^\epsilon(x) \phi(x) dx 
\leq \epsilon C \int_\Omega \parallel D^\beta \phi \parallel^2_{L^2} [F \phi]_{L^\infty} + \parallel D^\beta \phi \parallel [D(F \phi)]_{L^2}. \]
Since $\phi$ is arbitrary, we conclude the lemma.

Lemma 3.15. Let $n$ be as in lemma 3.12, then $n$ is a minimizing harmonic map in $H^2_{\gamma}(\Omega, N)$ and
\[
\int_\Omega a_{ij}^{\epsilon} \left( \frac{x}{\epsilon} \right) d^\alpha \eta dx 
\to a_0 \int_\Omega \parallel \nabla n \parallel^2 dx.
\]
Proof: To show that $n$ is actually a minimizing harmonic map subject to its boundary constraints, we need to introduce the correctors. Let $m_\epsilon$ be a cut-off function defined as follows
\[
m_\epsilon \in C(\Omega), \quad m_\epsilon(x) = 0 \quad \text{if } d(x, \partial \Omega) \leq \epsilon, \quad m_\epsilon(x) = 1 \quad \text{if } d(x, \partial \Omega) \geq 2\epsilon, \quad e^{(\gamma)} |D^\gamma m_\epsilon(x)| \leq c_\gamma, \forall \gamma \in \mathbb{N}.
\]
Here $c_\gamma$ depends on $\gamma$ but does not depend on $\epsilon$.

For fixed positive number $L$, we define $h^{\gamma}_{\beta} \in C^\infty(M^{k \times d}, M^{k \times d})$ by
\[
h^{\gamma}_{\beta}(y) = \begin{cases}
h^{\gamma}_{\beta} \text{ smooth} & \parallel y \parallel \leq L \\
0 & L < \parallel y \parallel < L + 1 \\
\parallel y \parallel \geq L + 1
\end{cases}
\]
with
\[
\int_{L < \parallel y \parallel < L + 1} \left( h^{\gamma}_{\beta}(y) \right)^2 dy < \frac{1}{L^2}.
\]

We consider
\[
\mu_\epsilon(n) = \{ -e m_\epsilon(x) \chi_{B}(x) \frac{\partial}{\partial y} \chi_{B}^{\epsilon}(Dn(x)) \}_{i=1}^k,
\]
where $\chi_{B}^{\epsilon} = \{ \chi_{B}^{\epsilon} \}$ is defined by (3.1). Let $w \in H^1_{\gamma}(\Omega, N)$ be a given function, when $\epsilon$ is small enough, $w + \mu_\epsilon(n)$ lies in a small neighborhood of $N$ on which the nearest point projection $\Pi$ is well defined, then
\[
w_{\epsilon} = \Pi \circ (w + \mu_\epsilon(n)) \in H^1_{\gamma}(\Omega, N)
\]
and we have
\[
X_{\epsilon}(w) = a_{ij} \left( \frac{x}{\epsilon} \right) D^\alpha w_i^\epsilon(x) \cdot D^\beta w_j^\epsilon(x) dx,
\]
where
\[
D^\alpha w_\epsilon^L(x) = D^\alpha w(x) - m_\epsilon(x)D^\alpha_{\gamma k} \left( \frac{x}{\epsilon} \right) \eta^k_{\beta}(Dw(x)) - r^\alpha_\epsilon(x)
\]

where
\[
r^\alpha_\epsilon(x) = \epsilon \left\{ d\Pi_{\alpha} \circ D^\alpha (m_\epsilon(x)\chi^\beta_{\epsilon} \left( \frac{x}{\epsilon} \right) \eta^\beta_{\beta}(Dw(x))) \right\} \\
+ \text{Hess}\Pi_{\alpha} \left( m_\epsilon(x)\chi_{\epsilon}^\beta \left( \frac{x}{\epsilon} \right) L_{\alpha \beta}(Dw(x)), D^\alpha w \right) \\
- m_\epsilon(x)D^\alpha_{\gamma k} \left( \frac{x}{\epsilon} \right) \eta^k_{\beta}(Dw(x)) + o(\epsilon).
\]

By virtue of the construction of \( m_\epsilon \) and properties of \( \chi_{\epsilon}^\beta \), \( 4\eta_{\epsilon}^k \), we have
\[
r^\alpha_\epsilon \rightarrow 0 \text{ in } L^2.
\]

Therefore, if we set
\[
A^\alpha (w) = D^\alpha w(x) - m_\epsilon(x)D^\alpha_{\gamma k} \left( \frac{x}{\epsilon} \right) \eta^k_{\beta}(Dw(x)),
\]

and let
\[
Y^L_\epsilon(w) = \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) A^\alpha(w) \cdot A^\beta(w),
\]

we have as \( \epsilon \rightarrow 0 \)
\[
X^L_\epsilon(w) - Y^L_\epsilon(w) \rightarrow 0 \quad \forall w \in H^1_\gamma(\Omega, N).
\]

But we can pass to the limit in (3.16); we obtain (here and in the following we always write \( a_{\alpha\beta}(x) = (a_{\alpha \beta}^{ij}(x)) \in M^{k \times k} \)
\[
\lim_{\epsilon \rightarrow 0} Y^L_\epsilon = \int_\Omega \left( a_{\alpha\beta}^{ij} D^\alpha w_i D^\beta w_j - \int_\Omega \left( a_{\alpha\beta}^{ij} D^\alpha_{\gamma k} \chi_{\epsilon}^{\alpha} \eta^k_{\beta}(Dw(x)) D^\alpha w_i \right) dx \\
- \int_\Omega \left( a_{\alpha\beta}^{ij} D^\alpha_{\gamma k} \chi_{\epsilon}^{\beta} \eta^k_{\gamma}(Dw(x)) D^\beta w_j \right) dx \right) + \int_\Omega \left( a_{\alpha\beta}^{ij} D^\alpha_{\gamma k} \chi_{\epsilon}^{\gamma} \chi_{\epsilon}^{\delta} \eta^k_{\delta}(Dw(x)) \right) \eta^k_{\gamma}(Dw(x)) dx.
\]

We then let \( L \rightarrow \infty \) in (3.17), by choice of \( b^\beta_{\gamma} \), we have
\[
\lim_{L \rightarrow \infty} \lim_{\epsilon \rightarrow 0} Y^L_\epsilon(w) = \int_\Omega p_{\alpha\beta}^{ij} D^\alpha w_i D^\beta w_j dx
\]

where
\[
p_{\alpha\beta}^{ij} = (a_{\alpha\beta}^{ij} - (a_{\gamma k}^{ij} D^\gamma_{\gamma k} \chi_{\epsilon}^{\alpha} \chi_{\epsilon}^{\beta}) - (a_{\alpha k}^{\delta} D^\beta_{\gamma j} \chi_{\epsilon}^{\gamma} \chi_{\epsilon}^{\delta}) + (a_{\gamma k}^{\delta} D^\gamma_{\gamma k} \chi_{\epsilon}^{\alpha} D^\beta_{\gamma j} \chi_{\epsilon}^{\beta}).
\]

Note
\[
p_{\alpha\beta}^{ij} = a_1(P^\alpha_i, P^\beta_j - \chi_{\epsilon}^\beta) = a_1(\chi_{\epsilon}^\beta - P^\beta_j, -P^\alpha_i),
\]

(3.18) then gives
\[
\lim_{L \rightarrow \infty} \lim_{\epsilon \rightarrow 0} Y^L_\epsilon(w) = a(w, w).
\]

From the assumption that \( n^\epsilon \) is a minimizer of
\[
\inf_{n \in M} a_{\epsilon}(n, n) = \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha n(x) \cdot D^\beta n(x) dx
\]
in $H^1_g(\Omega, N)$, we know for any $L$ fixed,
\[
\int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon(x) \cdot D^\beta n^\epsilon(x) dx \leq \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha w^\epsilon(x) \cdot D^\beta w^\epsilon(x) dx = X^L_\epsilon(w),
\]
passing to the limit,
\[
\limsup_{\epsilon \to 0} \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon(x) D^\beta n^\epsilon_j(x) dx \leq \lim_{\epsilon \to 0} X^L_\epsilon(w) = \lim_{\epsilon \to 0} Y^L_\epsilon(w)
\]
Let $L \to \infty$, using (3.19), we have
\[
\limsup_{\epsilon \to 0} \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon(x) D^\beta n^\epsilon_j(x) dx \leq a(w, w). \quad (3.20)
\]
On the other hand, we let $z_\epsilon = n^\epsilon - w^\epsilon_L$, we have
\[
0 \leq a^\epsilon(z^\epsilon, z^\epsilon) = a^\epsilon(n^\epsilon - w^\epsilon_L, n^\epsilon - w^\epsilon_L) = a^\epsilon(n^\epsilon, n^\epsilon) - 2a^\epsilon(n^\epsilon, w^\epsilon_L) + a^\epsilon(w^\epsilon_L, w^\epsilon_L). \quad (3.21)
\]
While from lemma 3.12 and lemma 3.14, we have
\[
a^\epsilon(n^\epsilon, w^\epsilon_L) = \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon(x) \cdot \left( D^\beta w(x) - m_\alpha(x) D^\alpha_g \chi_\epsilon^{\gamma} \left( \frac{x}{\epsilon} \right) h^\gamma(Dw(x)) \right) dx
= a^\epsilon(n^\epsilon, w^\epsilon_L) - \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) m_\alpha(x) D^\beta n^\epsilon_j(x) D^\alpha_g \chi_\epsilon^{\gamma} \left( \frac{x}{\epsilon} \right) h^\gamma(Dw(x)) dx
- \int_\Omega a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\beta n^\epsilon_j(x) r_\epsilon^\alpha dx \to a(n, w).
\]
(3.22)
Plug in $w = n$ to (3.22), together with (3.21) we have
\[
0 \leq \liminf_{\epsilon \to 0} a^\epsilon(n^\epsilon, n^\epsilon) - a(n, n). \quad (3.23)
\]
We then proved
\[
\lim_{\epsilon \to 0} a^\epsilon(n^\epsilon, n^\epsilon) = a(n, n).
\]
Finally it follows from (3.20), (3.23) and (3.3) that $n$ is a minimizing harmonic map in $H^1_g(\Omega, N)$.
In fact, we could prove the following local convergence lemma:

**Lemma 3.16.** Let $n^\epsilon$ be as in lemma 3.12, then there exists a subsequence $n^{\epsilon_k}$ and a minimizing harmonic map $n \in H^1_g(\Omega, N)$ such that for any $B_r(x) \subset \Omega$, we have
\[
\int_{B_r(x)} a_{\alpha\beta} \left( \frac{y}{\epsilon_k} \right) D^\alpha n^{\epsilon_k}(y) \cdot D^\beta n^{\epsilon_k}(y) dy \to \int_{B_r(x)} a_0 |\nabla n(y)|^2 dy.
\]
**Proof:** Since $n^\epsilon$ is bounded in $H^1_g(\Omega, N)$, we can find a subsequence $n^{\epsilon_k}$ and weakly harmonic map $n$ such that $n^{\epsilon_k} \rightharpoonup n$ in $H^1_g(\Omega, N)$. Let $m^\epsilon_r$ be a cut off function defined as follows
\[
m^\epsilon_r \in \mathcal{D}(B_r(x)),
m^\epsilon_r(y) = 0 \quad \text{if } d(y, \partial B_r(x)) \leq \epsilon,
m^\epsilon_r(y) = 1 \quad \text{if } d(y, \partial B_r(x)) \geq 2\epsilon,
\]
\[
e^{r\gamma} |D^\gamma m^\epsilon_r(y)| \leq c_\gamma, \forall \gamma \in \mathbb{N}, c_\gamma \text{ depends on } r, \gamma \text{ but not on } \epsilon. \quad (3.24)
\]
For $L$ fixed positive number, $L^g_{ij} \in C^{\infty}(M^{k \times d}, M^{k \times d})$ is defined by (3.15). For any $v \in W^{1,2}(B_r(x), N)$, we consider
\[ \mu^{L}_{it} (v) = \{ -\varepsilon m_t(x) x^i \frac{(x)}{\epsilon} L^g_{ij}(Dv(x)) \} . \]
For $\epsilon$ small enough, we can define
\[ v^{L}_{it} = 1 \circ (v + \mu^{L}_{it}(v)). \] (3.25)
Now follow the same proof as in lemma 3.15, we can prove that
\[ \lim_{L \to \infty} \lim_{\epsilon \to 0} \int_{B_r(x)} a_{ij} \left( \frac{y}{\epsilon} \right) D^\alpha u^L_{ij}(y) \cdot D^\beta v^L_{ij}(y) dy \to \int_{B_r(x)} a_0 \nabla u \cdot \nabla v dy. \] (3.26)
Let $a^L_{it}(n, m) = \int_{B_r(x)} a_{ij} \left( \frac{y}{\epsilon} \right) D^\alpha n^L_{ij}(y) \cdot D^\beta m^L_{ij}(y) dy$, $a_t(u, v) = \int_{B_r(z)} a_0 \nabla u \cdot \nabla v$. Take subsequence $n^{\epsilon, m}$ of $\{ n^{\epsilon, m} \}$ such that
\[ a^L_{it}(n^{\epsilon, m}, n^{L}) \to a_t(n^{\epsilon, m}, n) \text{ weakly in } L^2(B_r(x), M^{k \times d}), \]
\[ n^{\epsilon, m} \to n \text{ weakly in } W^{1,2}(B_r(x), N), \]
\[ a^L_{it}(n^{\epsilon, m}, n^{\epsilon, m}) \to \lim_{\epsilon \to 0} a^L_{it}(n^{\epsilon, m}, n^{\epsilon, m}). \] (3.27)
A similar argument as in lemma 3.12, we can show that
\[ (\xi^L_{ij}, D^\beta v_j) = \int_{B_r(x)} \xi^L_{ij} D^\beta v_j = a_t(u, v). \] (3.28)
Using (3.27) and (3.28), we can argue in the same way as in lemma 3.12 and lemma 3.14 to obtain
\[ \lim_{L \to \infty} \lim_{\epsilon \to 0} \int_{B_r(x)} a^L_{ij} \left( \frac{y}{\epsilon} \right) D^\alpha n^L_{ij}(y) D^\beta v^L_{ij}(y) dy \to \int_{B_r(x)} a_0 \nabla n \cdot \nabla v dy. \] (3.29)
On the other hand, we have
\[ 0 \leq a^L_{it}(n^{\epsilon, m} - n^{L})^2 \quad \text{for all } \quad \xi \in (0, 1), \quad \delta > 0. \]
By (3.26), (3.27) and (3.29), this implies
\[ \lim_{\epsilon \to 0} a^L_{it}(n^{\epsilon, m}, n^{\epsilon, m}) \geq a_t(n, n). \]
To prove
\[ \lim_{\epsilon \to 0} a^L_{it}(n^{\epsilon, m}, n^{\epsilon, m}) \leq a_t(n, n), \] (3.31)
we need to modify the argument in lemma 3.15. Since now $n^\epsilon$ does not have the same boundary condition on $\partial B_r(x)$, we need to apply Luckhause’s lemma (2.7) to construct suitable comparison functions. Let $B_{r_0}(x) \subset \Omega$ and let $\theta \in (0, 1)$, $\delta > 0$ be given. Choose any $M \in \mathbb{N}$ with $\limsup_{l \to \infty} E_{\epsilon, l}(n^{\epsilon, m}, r_0, x) < M \delta$ and note that if $\varepsilon \in (0, 1 - \theta/M)$ we must have some integer $l \in \{ 2, \cdots, M \}$ such that
\[ r_0^{-2-d} \int_{B_{r_0}(\theta l + 1)(x) \setminus B_{r_0}(\theta l - 1) (x)} a^L_{ij} \left( \frac{y}{\epsilon} \right) D^\alpha n^{\epsilon, m}_L(y) D^\beta n^{\epsilon, m}_L(y) dy < \delta \]
for infinitely many $\epsilon, l$, because otherwise we get that $E_{\epsilon, l}(n^{\epsilon, m}, r_0, x) > M \delta$ for all sufficiently large $l$. Thus
choose such an \( l \), letting \( r = r_0(\theta + (l - 2)\varepsilon) \) and noting that
\[ r(1 + \varepsilon) \leq r_0(\theta + l\varepsilon) < r_0, \quad r \in (\theta r_0, r_0) \]
such that
\[
\frac{r^2}{\varepsilon} \int_{B_{r(1+\varepsilon)}(y) \setminus B_r(x)} a_{ij} \left( \frac{y}{\varepsilon_k} \right) D^\alpha n_{\varepsilon_k}^i(y) D^\beta n_{\varepsilon_k}^j(y) dy \leq \delta
\]
for some subsequence \( n_{\varepsilon_k} \) (for simplicity of notation, we shall denote the
subsequence by \( n_{\varepsilon_k} \) from now on). Passing to the limit, we then have
\[
\frac{r^2}{\varepsilon} \int_{B_{r(1+\varepsilon)}(y) \setminus B_r(x)} |\nabla n|^2 dx \leq \delta.
\]
By lemma 2.7, we can find \( w_{\varepsilon_k} \in W^{1,2}(B_{r(1+\varepsilon)}(y) \setminus B_r(y); N) \) such that \( w_{\varepsilon_k} = n \) in
a neighborhood of \( \partial B_r(x) \), \( w_{\varepsilon_k} = n_{\varepsilon_k} \) in a neighborhood of \( \partial B_{r(1+\varepsilon)}(x) \) and
\[
\frac{r^2}{\varepsilon} \int_{B_{r(1+\varepsilon)}(y) \setminus B_r(x)} |\nabla w_{\varepsilon_k}|^2 dx \\
\leq C \left( \int_{B_{r(1+\varepsilon)}(y) \setminus B_r(x)} (|\nabla n|^2 + |\nabla n_{\varepsilon_k}|^2 + \varepsilon_k^{-2} r^{-2} |n - n_{\varepsilon_k}|^2) dx, \right)
\]
where \( C \) depends only on \( d, N \). Now consider \( n_{L_{\varepsilon_k}}(x) \) defined by formula (3.25), i.e.
\[
n_{L_{\varepsilon_k}} = \Pi \circ (n + \mu_{\varepsilon_k}^L(n))
\]
and let
\[
\tilde{n}_{\varepsilon_k} = \begin{cases} 
  n_{\varepsilon_k} & \text{if } B_{r_0}(y) \setminus B_{r(1+\varepsilon)}(y) \\
  w_{\varepsilon_k} & \text{if } B_{r(1+\varepsilon)}(x) \setminus B_r(x) \\
  n_{L_{\varepsilon_k}} & \text{if } B_r(x)
\end{cases}
\]
Then by minimality of \( n_{\varepsilon_k} \) we have
\[
\int_{B_{r(1+\varepsilon)}(y) \setminus B_r(x)} a_{ij} \left( \frac{y}{\varepsilon_k} \right) D^\alpha n_{\varepsilon_k}^i(y) \cdot D^\beta n_{\varepsilon_k}^j(y) dy \\
\leq \int_{B_{r(1+\varepsilon)}(y) \setminus B_r(x)} a_{ij} \left( \frac{y}{\varepsilon_k} \right) D^\alpha \tilde{n}_{\varepsilon_k}^i(y) \cdot D^\beta \tilde{n}_{\varepsilon_k}^j(y) \\
\leq \int_{B_r(x)} a_{ij} \left( \frac{y}{\varepsilon_k} \right) D^\alpha n_{L_{\varepsilon_k}}^i(x) \cdot D^\beta n_{L_{\varepsilon_k}}^j(x) dy + 2 \int_{B_{r(1+\varepsilon)}(y) \setminus B_r(x)} |\nabla w_{\varepsilon_k}|^2 dx.
\]
(3.32)
By (3.26), taking limit in (3.32) gives
\[
\liminf_{l \to \infty} \int_{B_r(x)} a_{ij} \left( \frac{x}{\varepsilon_l} \right) D^\alpha n_{\varepsilon_l}^i(x) \cdot D^\beta n_{\varepsilon_l}^j(x) dx \leq \int_{B_r(x)} a_0 |\nabla n|^2 dx + C\delta.
\]
Since \( \delta \) is arbitrary, (3.31) follows. Thus we can find a subsequence such that
\[
a_{ij}^L(n_{\varepsilon_l}, n_{\varepsilon_l}) \to a_r(n, n). \square
\]
In fact, the above argument actually proves the following statement:

**Lemma 3.17.** Assume sequence \( n_{\varepsilon_l} \to n \) in \( H^1_0(\Omega, \mathbb{R}^k) \), where \( n_{\varepsilon_l} \) is a minimizer
of \( I_\varepsilon \) in \( H^1_0(\Omega, N) \). Then for \( B_r(x) \subset \Omega \), we have
\[
\int_{B_r(x)} a_{ij} \left( \frac{y}{\varepsilon_j} \right) D^\alpha n_{\varepsilon_l}^i(y) \cdot D^\beta n_{\varepsilon_l}^j(y) dy \to a_0 \int_{B_r(x)} |\nabla n|^2 dy
\]
and we can find a subsequence $n'^{j_k}$ such that

$$\int_{B_r(x)} |D^\alpha n'^{j_k} - D^\alpha n_i - D^\alpha \chi_{\xi_j}(\frac{x}{\varepsilon_j}) \frac{\partial n'^{j_k}}{\partial x^\beta}|^2 \to 0.$$ (3.33)

**Proof:** Let $n'^{j_k}$ be such that

$$\int_{B_r(x)} a_{\alpha\beta} \left( \frac{x}{\varepsilon_j} \right) D^\alpha n'^{j_k} \cdot D^\beta n'^{j_k} \to \liminf_{j \to \infty} \int_{B_r(x)} a_{\alpha\beta} \left( \frac{x}{\varepsilon_j} \right) D^\alpha n'^{j} \cdot D^\beta n'. $$

The previous lemma showed that

$$\int_{B_r(x)} a_{\alpha\beta} \left( \frac{x}{\varepsilon_j} \right) D^\alpha n'^{j_k} \cdot D^\beta n'^{j_k} \to \int_{B_r(x)} a_0 |\nabla n|^2.$$ Moreover, for each $L > 0$ fixed, we have

$$\int_{B_r(x)} \left| D^\alpha n'^{j_k} - D^\alpha n_i - m_{\varepsilon_j}(x) D^\alpha \chi_{\xi_j}(\frac{x}{\varepsilon_j}) \frac{\partial n'^{j_k}}{\partial x^\beta} \right|^2 \leq \int_{B_r(x)} a_{\alpha\beta} \left( \frac{x}{\varepsilon_j} \right) \left( D^\alpha n'^{j_k} - m_{\varepsilon_j}(x) D^\alpha \chi_{\xi_j}(\frac{x}{\varepsilon_j}) \frac{\partial n'^{j_k}}{\partial x^\beta} \right) \left( D^\beta n'^{j_k} - m_{\varepsilon_j}(x) D^\beta \chi_{\xi_j}(\frac{x}{\varepsilon_j}) \frac{\partial n'^{j_k}}{\partial x^\beta} \right) \eta_j^L(Dn(x)).$$ (3.34)

Let $\varepsilon_j \to 0$, then $L \to \infty$, the right hand side of (3.34) converges to \(\liminf_{j \to \infty} a_{\alpha\beta}(n'^{j}, n'^{j}) - a_\varepsilon(n, n) = 0\), (3.33) follows.

### 3.2. Hölder estimate

In this and next section, we prove some uniform small energy estimates on $n'$. More precisely, we have

**Theorem 3.6.** There exists a constant $\delta_0$ independent of $\varepsilon$ such that for any $B_r(x) \in \Omega$, and any minimizer $n'$ of $I_\varepsilon$, satisfying

$$E_\varepsilon(n', r, x) = \frac{1}{r^{n+2}} \int_{B_r(x)} a_{\alpha\beta}^{ij} \left( \frac{x}{\varepsilon} \right) D^\alpha n'_i D^\beta n'_j dx \leq \delta_0$$

then $n' \in C^{\beta}(\bar{B}_\frac{r}{2}(x))$ for all $\beta < 1$.  

We prove the theorem following the compactness argument developed by Avellenda and Lin for linear elliptic system (See [AL1, AL2, AL3]). Namely, we prove the uniform Hölder estimate in three steps.

**Step 1.** Show that there exist constants $\theta \in (0, 1), \mu \in (0, 1), \varepsilon_0, \delta_0$ depend only on $d, N$ such that if

$$E_\varepsilon(n', 1, 0) \leq \delta_0,$$

then for $\varepsilon \leq \varepsilon_0$,

$$E_\varepsilon(n', \theta, 0) \leq \theta^{2\mu} E_\varepsilon(n', 1, 0).$$

This step follows directly from the small energy estimates for minimizing harmonic maps and the strong convergence results of $E_\varepsilon(n', r, 0)$ by lemma 3.17.

**Step 2.** A recursive argument of the step 1 implies

$$E_\varepsilon(n', r, 0) \leq r^{2\mu} E_\varepsilon(n', 1, 0)$$

for all $r \geq \frac{1}{\varepsilon_0}$.

**Step 3.** Blow up argument in $\varepsilon$ scale.
Before we present the lemmas, we specify that from now on, by a minimizer of $I(n) = \int_B a_{\alpha\beta} D^{\alpha\beta} n \cdot D^{\beta} n$ we mean $I(n) \leq I(m)$ for any $m \in H^1(B, N)$ with $m - n$ compactly supported in $B$.

**Lemma 3.18.** For any $0 < \mu < 1$, there exist $\theta, 0 < \theta < 1$, and $\epsilon_0, \delta_0 > 0$ depending only on $d$ and $N$, such that if $n^\epsilon$ is a minimizer of $I_{\epsilon} = \int_{B_{\epsilon}(0)} a_{ij}^{\epsilon} (\frac{x}{\epsilon}) D^{ij} n_{\epsilon}^i (x) D^{\epsilon j} n_{\epsilon}^j (x) dx$ with

$$\int_{B_{\epsilon}(0)} a_{ij}^{\epsilon} (\frac{x}{\epsilon}) D^{\alpha\beta} n_{\epsilon}^i (x) D^{\beta j} n_{\epsilon}^j (x) dx \leq \delta_0,$$

then for all $\epsilon \leq \epsilon_0$,

$$\frac{1}{\theta^{d/2}} \int_{B_{\epsilon}(0)} a_{ij}^{\epsilon} (\frac{x}{\epsilon}) D^{\alpha\beta} n_{\epsilon}^i (x) D^{\beta j} n_{\epsilon}^j (x) dx \leq \theta^{2\mu} \int_{B_{\epsilon}(0)} a_{ij}^{\epsilon} (\frac{x}{\epsilon}) D^{\alpha\beta} n_{\epsilon}^i (x) D^{\beta j} n_{\epsilon}^j (x) dx.$$

(3.35)

**Proof:** Suppose $\mu < \mu' < 1$. Were (3.35) false, then for any fixed $\theta \in (0, 1)$, $\delta > 0$ which will be chosen later, we could find minimizers $n^{\epsilon_k}$ of $I_{\epsilon_k}$ satisfying

$$\int_{B_{\epsilon}(0)} a_{ij}^{\epsilon_k} (\frac{x}{\epsilon_k}) D^{\alpha\beta} n_{\epsilon_k}^i (x) D^{\beta j} n_{\epsilon_k}^j (x) \leq \delta,$$

yet

$$\frac{1}{\theta^{d/2}} \int_{B_{\epsilon}(0)} a_{ij}^{\epsilon_k} (\frac{x}{\epsilon_k}) D^{\alpha\beta} n_{\epsilon_k}^i (x) D^{\beta j} n_{\epsilon_k}^j (x) dx > \theta^{2\mu} \int_{B_{\epsilon}(0)} a_{ij}^{\epsilon_k} (\frac{x}{\epsilon_k}) D^{\alpha\beta} n_{\epsilon_k}^i (x) D^{\beta j} n_{\epsilon_k}^j (x) dx.$$

(3.36)

By the homogenization limit lemmas 3.12 and 3.15, we know there exists a subsequence (for simplicity, we denote by $n^{k}$) such that $n^{k}$ is a minimizer of $I_{\epsilon_k}$ and $n^{k} \rightarrow n$ where $n$ is a minimizing harmonic map and

$$\int_{B_{\epsilon}(0)} a_{ij}^{\epsilon_k} (\frac{x}{\epsilon_k}) D^{\alpha\beta} n_{\epsilon_k}^i (x) D^{\beta j} n_{\epsilon_k}^j (x) dx \rightarrow \int_{B_{\epsilon}(0)} |\nabla n|^2 dx,$$

$$\int_{B_{\epsilon}(0)} a_{ij}^{\epsilon_k} (\frac{x}{\epsilon_k}) D^{\alpha\beta} n_{\epsilon_k}^i (x) D^{\beta j} n_{\epsilon_k}^j (x) dx \rightarrow \int_{B_{\epsilon}(0)} |\nabla n|^2 dx.$$

Since $n$ is a minimizing harmonic map, there exists a constant $\delta_0 > 0$, such that if

$$\int_{B_{\epsilon}(0)} |\nabla n|^2 \leq \delta_0$$

then for $\theta$ small enough, the following holds

$$\frac{1}{\theta^{d/2}} \int_{B_{\epsilon}(0)} |\nabla n|^2 dx \leq \theta^{2\mu} \int_{B_{\epsilon}(0)} |\nabla n|^2 dx.$$

Now take $\delta = \frac{\delta_0}{2}$, pass to the limit in (3.36), a contradiction arises.

**Lemma 3.19.** Given $\mu, 0 < \mu < 1$, let $\theta, \epsilon_0, \delta_0$ be as in lemma 3.18. Then for all $n^\epsilon$, $n^\epsilon$ being a minimizer of $I_{\epsilon}$, satisfying

$$E_{\epsilon}(n^\epsilon, 1, 0) \leq \delta_0,$$

for all $k$ such that $\epsilon/\theta^k \leq \epsilon_0$, we have

$$E_{\epsilon}(n^\epsilon, \theta^k, 0) \leq \theta^{2k\mu} E_{\epsilon}(n^\epsilon, 1, 0).$$

(3.37)
Lemma 3.20. Suppose $B$. Note that to repeat the above recursive argument for any fixed ball $R^k$.

Remark 3.4. Then $w^\epsilon \in H^1(B_1(0), N)$ and from (3.37)

$$\int_{B_1(0)} a_{\alpha\beta} \left( \frac{\theta k z}{\epsilon} \right) D^\alpha w^\epsilon(z) \cdot D^\beta w^\epsilon(z) dz = \frac{1}{\theta^{(d-2)k}} \int_{B_{\epsilon k}(0)} a_{\alpha\beta} \left( \frac{y}{\epsilon} \right) D^\alpha n^\epsilon(y) \cdot D^\beta n^\epsilon(y) dy \leq \delta_0.$$  

and $w^\epsilon$ is a minimizer of $\int_{B_2(0)} a_{\alpha\beta} \left( \frac{\theta k z}{\epsilon} \right) D^\alpha n(z) \cdot D^\beta n(z) dz$. Apply lemma 3.18 to $w^\epsilon$, we obtain

$$\frac{1}{\theta^{(d-2)(k+1)}} \int_{B_{\epsilon k+1}(0)} a_{ij} \left( \frac{\theta k z}{\epsilon} \right) D^\alpha n^\epsilon_i(y) D^\beta n^\epsilon_j(y) dy \leq \theta^{2(k+1)} \int_{B_1(0)} a_{ij} \left( \frac{0}{\epsilon} \right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x) dx.$$  

Rewriting (3.39) using (3.38) we see that

$$\frac{1}{\theta^{(d-2)(k+1)}} \int_{B_{\epsilon k+1}(0)} a_{ij} \left( \frac{0}{\epsilon} \right) D^\alpha n^\epsilon_i(y) D^\beta n^\epsilon_j(y) dy \leq \theta^{2(k+1)} \int_{B_1(0)} a_{ij} \left( \frac{0}{\epsilon} \right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x) dx.$$  

RemarK 3.4. Note that to repeat the above recursive argument for any fixed ball $B(x,r) \subset \Omega$, we actually need modify lemma 3.18 into following version:

Lemma 3.20. Suppose $n^\epsilon$ is a minimizer of $\int_{B_1(0)} a_{\alpha\beta} \left( \frac{x+x_0}{\epsilon} \right) D^\alpha n \cdot D^\beta n$, $x_0$ is a fixed point in $\mathbb{R}^d$. Then we can find $\delta_0$ independent of $x_0, n^\epsilon$, such that if

$$\int_{B_1(0)} a_{\alpha\beta} \left( \frac{x+x_0}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \delta_0,$$

then

$$\frac{1}{\theta^{d-2}} \int_{B_0(0)} a_{\alpha\beta} \left( \frac{x+x_0}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \theta^{2d} \int_{B_1(0)} a_{\alpha\beta} \left( \frac{x+x_0}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon.$$

Proof: The proof amounts to a strong convergence of the corresponding energy independent of base point $x_0$. For this purpose, we need only to modify the correctors by the same translation. i.e. we choose correctors by $m^\epsilon(x+x_0)D_\alpha \chi \left( \frac{x+x_0}{\epsilon} \right) \nabla n(x)$, then we obtain the same energy convergence results.

The rest is similar. The same argument applies to the recursive argument for Lipschitz estimate in the next section.

The next lemma constitutes a priori interior Hölder estimate for minimizers of $\int a_{\alpha\beta} \left( \frac{x}{\epsilon} \right) D^\alpha n \cdot D^\beta n$. For simplicity, we state it for minimizers on $B_1(0)$, the most general case will follow by localization and scaling arguments.

Lemma 3.21. Given $\mu, 0 < \mu < 1$, there exists $\delta_0 > 0$ depending only on $d, N$, such that if $n^\epsilon$ is a minimizer of $\int_{B_2(0)} a_{\alpha\beta} \left( \frac{z}{\epsilon} \right) D^\alpha n_i D^\beta n_j$ satisfying

$$\int_{B_1(0)} a_{\alpha\beta} \left( \frac{z}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \delta_0,$$
then there exists a constant $C$ depending only on $d, N, \mu$ such that
\[ [n']_{C^{0,\nu}(B_\frac 12(0))} \leq C \int_{B_1(0)} a_{\alpha\beta} \left( \frac x\epsilon \right) D^\alpha n'(x) \cdot D^\beta n'(x) dx. \]

**Proof:** We denote by $C$ a generic constant depending on $d, N, \mu$ possibly changing from one estimate to another. From lemma 3.19, we conclude that for all $r \geq \epsilon/\epsilon_0$,
\[ \frac 1{\epsilon^{d-2}} \int_{B_r(0)} a_{\alpha\beta} \left( \frac x\epsilon \right) D^\alpha n'(x) \cdot D^\beta n'(x) \leq C \epsilon^{2\mu} \int_{B_1(0)} a_{\alpha\beta} \left( \frac x\epsilon \right) D^\alpha n'(x) \cdot D^\beta n'(x). \]

We take $r = 2\epsilon/\epsilon_0$ in (3.40) and define the new function
\[ w'(x) = n'(\epsilon x) \quad x \in B_{\frac 2\epsilon}(0). \]

Then $w'$ is a minimizer of $I_1 = \int_{B_{\frac 2\epsilon}(0)} a_{\alpha\beta}(x) D^\alpha n \cdot D^\beta n$. From the small energy estimates in section 2, we conclude that there exists a $\delta_1 > 0$, such that if
\[ \int_{B_{\frac 2\epsilon}(0)} a_{\alpha\beta}(x) D^\alpha w' \cdot D^\beta w' \leq \delta_1, \]
then
\[ \sup_{|x| < \frac 1\epsilon} \sup_{0 < r < \frac 1\epsilon} \frac 1{r^{d-2+2\mu}} \int_{B_r(x)} a_{\alpha\beta}(x) D^\alpha w'(x) \cdot D^\beta w'(x) \]
\[ \leq C \epsilon^{1-2} \int_{B_{\frac 2\epsilon}(0)} a_{\alpha\beta}(x) D^\alpha w'(x) \cdot D^\beta w'(x) dx. \]

Setting $s = \epsilon r$, plug (3.2) into (3.42) we see that
\[ \sup_{|x| < \frac 1\epsilon} \sup_{0 < s < \frac 1\epsilon} \frac 1{s^{d-2+2\mu}} \int_{B_s(x)} a_{\alpha\beta} \left( \frac x\epsilon \right) D^\alpha n' \cdot D^\beta n' \]
\[ \leq C \int_{B_{\frac 2\epsilon}(0)} a_{\alpha\beta} \left( \frac x\epsilon \right) D^\alpha n'(x) \cdot D^\beta n'(x). \]

If we combine (3.40) and (3.43) and small energy estimates from theorem 2.1, the conclusion follows for all $\epsilon$.

**Remark 3.5.** It can be checked that when $a_{\alpha\beta}$ is bounded measurable, we still have the strong convergence of energy and the homogenization limit is a minimizing harmonic map. We thus conclude that the above uniform Hölder estimates holds for general case.

In fact, if we have the monotonicity formula or assume $N$ is simply connected, we can prove the following interesting lemma from the uniform Hölder estimates.

**Corollary 3.1.** (Singular points converge to singular points). Suppose $n'$ is a sequence of minimizers of $I_1 = \int_{\Omega} a_{\alpha\beta} D^\alpha n \cdot D^\beta n$ in $H_2^1(\Omega, N)$ converges weakly to $n$ in $H_2^1(\Omega, N)$. Assume $N$ is simply connected or monotonicity formula holds for $n'$ with a uniform constant, then

1. If $y'$ is a singular point for $n'$ such that $y' \to y \in \Omega$, then $y$ is a singular point for $n$. 

\[ \text{[n']}_{C^{0,\nu}(B_\frac 12(0))} \leq C \int_{B_1(0)} a_{\alpha\beta} \left( \frac x\epsilon \right) D^\alpha n'(x) \cdot D^\beta n'(x) dx. \]
(2) If \( y \in \Omega \) is a singular point for \( n \), then for all sufficiently small \( \epsilon \), \( n^\epsilon \) has a singular point at some \( y^\epsilon \) with \( y^\epsilon \rightarrow y \).

**Proof:** Consider (1). By previous results, we know \( n \) is a minimizing harmonic map in \( H^1_0(\Omega, n) \). If \( y \) is not a singular point of \( n \), then for \( r > 0 \) small enough, we have

\[
\frac{1}{r^{d-2}} \int_{B_r(y)} a_0|\nabla n|^2 \leq \frac{\delta_0}{2^d}.
\]

Here \( \delta_0 \) is given by lemma 3.21. By energy convergence result, we know for \( \epsilon \) small enough,

\[
\frac{1}{r^{d-2}} \int_{B_r(y)} a_\alpha\beta \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \frac{\delta_0}{2^{d-2}}.
\]

On the other hand, for \( \epsilon \) small enough, we have \( y^\epsilon \in B_\delta(y) \), hence

\[
\frac{1}{(\epsilon^d)^{d-2}} \int_{B_{\epsilon^2}(y^\epsilon)} a_\alpha\beta \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \frac{1}{r^{d-2}} \int_{B_r(y)} a_\alpha\beta \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \delta_0,
\]

by lemma 3.21, \( y^\epsilon \) is a regular point of \( n^\epsilon \), a contradiction.

With regard to (2). If the conclusion were false, we could find a \( r \) and a subsequence \( n^{\epsilon_k} \) such that there are no singular points of \( n^{\epsilon_k} \) in \( B_r(y) \). Without loss of generality, we may assume \( y = 0 \). From Lipschitz estimates lemma 2.6 plus the assumption that monotonicity formula holds or \( N \) is simply connected, one obtains a uniform bound on \( |\nabla n^{\epsilon_k}|_{L^\infty(B_{\frac{r}{2}}(0))} \). For any \( \delta > 0 \), when \( r \) small enough, we have \( \frac{1}{r^d} \int_{B_r(0)} a_\alpha\beta \left( \frac{x}{\epsilon} \right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \delta \). From strong convergence of the energy, we conclude \( \frac{1}{r^d} \int_{B_r(0)} a_0|\nabla n|^2 \leq \delta \). When \( \epsilon \) is small enough, this implies 0 is a regular point of \( n \). A contradiction. \( \square \)

In fact, if we assume \( a_\alpha\beta(x) \) to be continuous, we can consider the homogenization problem

\[
\min \int_{\Omega} a_\alpha\beta \left( \frac{x}{\epsilon} \right) D^\alpha n \cdot D^\beta n
\]

and study the asymptotic behavior of minimizer \( n^\epsilon \) as \( \epsilon \) approaches zero. One can check easily that the theorem 3.6 continues to hold in this case. Moreover, \( n^\epsilon \) converges weakly to a minimizing harmonic map \( n \in H^1(\Omega, N) \). Recall from Schoen and Uhlenbeck’s result (see [SU1]) the singular set of \( n^\epsilon \) is of dimension \( d - 3 \), in particular, when \( d = 3 \), the singular set of \( n^\epsilon \) is discrete. Later Almgren and Lieb ([AL]) obtained a uniform bound (depends only on geometry of \( \Omega \) and the energy of the boundary function of \( n \)) for the energy of singular points of a minimizing harmonic map \( n \) from \( \Omega \subset \mathbb{R}^3 \) onto \( S^2 \). In particular, they have the following theorem on uniform distance between singular points:

**Theorem 3.7.** ([AL]) **Theorem 2.1** Suppose \( n \) is a minimizing harmonic map from \( \Omega \subset \mathbb{R}^3 \) into \( S^2 \) having a singularity at \( y \in \Omega \). Let \( D \) denote the distance from \( y \) to \( \partial \Omega \). Then there is a universal constant \( C \) independent of \( \Omega, n, D, y \) etc. such that there is no other singularity within distance \( CD \) of \( y \).

As an application of our uniform small energy estimates, we obtain the following theorem on the uniform bound for the number of singular points of \( n^\epsilon \).

**Theorem 3.8.** Let \( a_\alpha\beta(x) \) be continuous, we consider the homogenization (3.44) for \( N = S^2 \) when \( d = 3 \). Then the total number of singular points \( N_\epsilon \) of \( n^\epsilon \) is bounded above by some \( l \) independent of \( \epsilon \).
Proof: Choose a subsequence $n^{\epsilon_k}$ which converges weakly to $n \in H^1_0(\Omega, S^2)$, then $n$ is a minimizing harmonic map and the singular points of $n^{\epsilon_k}$ converges to singular point of $n$. Note singular points of $n$ are isolated and the total number of singular points is bounded above by a constant depending only on $d, g$ and the geometry of $\Omega$. Let $p$ be a singular point of $n$, then there exists $r$ depending only on $d, g, \Omega$ such that there is no other singular point of $n$ in $B_r(p)$. Without loss of generality, we assume $p = 0$. Since singular points of $n$ are limits of $n^{\epsilon_k}$, we can always find singular points $p_k$ of $n^{\epsilon_k}$ such that $p_k \to 0$. We claim there exists a $L > 0$ such that for $k$ large enough, all singular points of $n^{\epsilon_k}$ close to 0 lie in $B(0, L\epsilon_k)$. Otherwise for a subsequence (we still denote by $n^{\epsilon_k}$), we can always find a singular point $p_k$ of $n^{\epsilon_k}$ with $|p_k| = \delta_k \to 0$ and $\frac{\delta_k}{\epsilon_k} \to 0$. Choose $\delta > 0$, we consider $w^{\epsilon_k} = n^{\epsilon_k}(\frac{\delta_k x}{\epsilon_k})$. Then $w^{\epsilon_k}$ is a minimizer of $\int_{B_1(0)} a_{\alpha\beta} \frac{\delta_k x}{\epsilon_k} D^\alpha w^{\epsilon_k} \cdot D^\beta w^{\epsilon_k}$ with
\[
\int_{B_1(0)} |\nabla w^{\epsilon_k}|^2 \leq C.
\]
The bound follows from the energy bound in lemma 2.10. Since $\frac{\delta_k}{\epsilon_k} \to 0$, we can argue as before and show that up to a subsequence $w^{\epsilon_k}$ converges weakly to a minimizing harmonic map $w$ and
\[
\int_{B_{\lambda}(0)} a_{\alpha\beta} \frac{\delta_k x}{\epsilon_k} D^\alpha w^{\epsilon_k} \cdot D^\beta w^{\epsilon_k} \to \int_{B_{\lambda}(0)} a_0 |\nabla w|^2 \quad (3.45)
\]
for any $r \leq \frac{\lambda}{2}$. Note each $w^{\epsilon_k}$ has a singular point $q_k$ on $\partial B_0(0)$. By corollary 3.1, we know $q_k$ converges to a singular point $q$ of $w$. On the other hand, we note 0 is also a singular point of $w$. In fact, if 0 is a regular point of $w$, then for some $r$ small enough, we have
\[
r^{2-d} \int_{B_r(0)} a_0 |\nabla w|^2 \leq \frac{1}{2} \delta_0.
\]
Here $\delta_0$ is a small constant as in lemma 3.21. By strong convergence of energy (3.45), we conclude that for $k$ large enough, we have
\[
r^{2-d} \int_{B_r(0)} a \left( \frac{\delta_k x}{\epsilon_k} \right) \left| \nabla w^{\epsilon_k} \right|^2 \leq \delta_0,
\]
which implies that
\[
\left( \frac{r \delta_k}{\delta} \right)^{2-d} \int_{B_{r\delta_k}(0)} a \left( \frac{x}{\epsilon_k} \right) \left| \nabla n^{\epsilon_k} \right|^2 \leq \delta_0.
\]
By uniform energy estimates lemma 3.21 and corollary 3.1, we conclude 0 is a regular point of $n$, a contradiction to our choice of 0. Therefore $w$ has a singular point at 0 and $\partial B_0(0)$. Since $w$ is a minimizing harmonic map from $B_1(0)$ into $S^2$, for any singular point $p$ of $w$ lies in $B_1(0)$, we conclude from theorem 3.7 there exists a $r$ independent of $w$, such that there are no other singular points of $w$ in $B_r(p)$. If we take $\delta$ small enough, that would be a contradiction. Therefore there exists a $L$ such that all singular points of $n^{\epsilon_k}$ close to 0 lies in $B_{L\epsilon_k}(0)$ for $k$ large enough.

The conclusion of the theorem now follows easily. In fact, modify the proof of theorem 2.1 in [AL] slightly, one can show that at $\epsilon$ scale the distance between the singular points of $n^\epsilon$ is given by $C\epsilon$ with $C$ independent of $n^\epsilon$. Hence there are $M$ singular points of $n^\epsilon$ in $B_{L\epsilon}(p)$ for each singular point $p$ of $n$, with $M$ independent
of \( \epsilon \). Since there are \( N \) singular points, there are at most \( MN \) singular points of \( n^\epsilon \) in \( \Omega \) with \( MN \) independent of \( \epsilon \). □

3.3. Gradient Estimates. In this section, we use the three step compactness method to prove \( L^\infty \) estimates on gradients of minimizers. In this section, \( \chi_{k_i} \) always denote the corrector defined in (3.1). \( m^\epsilon(x) \) is defined as in (3.24).

**Lemma 3.22.** Given \( \mu \in (0, 1) \), we can find \( \theta \in (0, 1) \) \( \delta_0, \epsilon_0 > 0 \) depending only on \( d, N, \mu \) such that the following statement is true: If \( n^\epsilon \) is a minimizer of \( \int_{B_1(0)} a(\frac{x}{\epsilon})|\nabla n|^2dx \) with

\[
\int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n|^2 \leq \delta_0,
\]

then for \( \epsilon \leq \epsilon_0 \), we have

\[
\frac{1}{\theta^2-1} \int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n|^2dx \leq \theta^{2\mu} \int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n|^2dx. \tag{3.46}
\]

and

\[
\int_{B_2(0)} |\nabla n^\epsilon(x) - A^\epsilon_\theta(x) - \nabla n^\epsilon B_2|^2dx \leq \theta^{2\mu} \int_{B_2(0)} |\nabla n^\epsilon(x) - A^\epsilon_\theta(x) - \nabla n^\epsilon B_2|^2dx + \theta^{2\mu}, \tag{3.47}
\]

where \( A^\epsilon_\theta = (A^\epsilon_{\chi_{k_i}}) \in M^{k \times d} \), \( A^\epsilon_{\chi_{k_i}} = D^2_y \chi_{k_i}(\frac{x}{\epsilon}) \frac{\partial n}{\partial x} B_\mu \).

**Proof:** (3.46) follows from lemma 3.18. We prove (3.47). If (3.47) fails, then for any fixed \( \mu, \theta \in (0, 1), \delta > 0 \) which will be chosen later, there would exist \( \epsilon_k \downarrow 0 \) and \( n^{\epsilon_k} \) such that \( n^{\epsilon_k} \) is a minimizer of \( \int_{B_1(0)} a(\frac{x}{\epsilon_k})|\nabla n|^2dx \) with

\[
\int_{B_1(0)} a\left(\frac{x}{\epsilon_k}\right)|\nabla n^{\epsilon_k}|^2 \leq \delta,
\]

but

\[
\int_{B_2(0)} |\nabla n^{\epsilon_k} - A^{\epsilon_k}_\theta - \nabla n^{\epsilon_k} B_2|^2dx > \theta^{2\mu} \int_{B_2(0)} |\nabla n^{\epsilon_k} - A^{\epsilon_k}_\theta - \nabla n^{\epsilon_k} B_2|^2dx + \theta^{2\mu}. \tag{3.48}
\]

From lemma 3.12 and lemma 3.17, we can find a subsequence (we still denote by \( n^{\epsilon_k} \)) and a minimizing harmonic map \( n \in H^1(B_2(0), N) \) such that

\[
n^{\epsilon_k} \rightharpoonup n \text{ weakly in } W^{1,2}(B_2(0))
\]

and

\[
\int_{B_1(0)} a\left(\frac{x}{\epsilon_k}\right)|\nabla n^{\epsilon_k}|^2 \rightharpoonup \int_{B_1(0)} a_0|\nabla n|^2,
\]

\[
\int_{B_2(0)} a\left(\frac{x}{\epsilon_k}\right)|\nabla n^{\epsilon_k}|^2 \rightharpoonup \int_{B_2(0)} a_0|\nabla n|^2,
\]

\[
\int_{B_2(0)} |\nabla n^{\epsilon_k} - \nabla n - \nabla y \chi(\frac{x}{\epsilon})|\nabla n|^2 \rightharpoonup 0,
\]

\[
\int_{B_2(0)} |\nabla n^{\epsilon_k} - \nabla n - \nabla y \chi(\frac{x}{\epsilon})|\nabla n|^2 \rightharpoonup 0.
\]
On the other hand, from the small energy estimates for minimizing harmonic maps, we know there exists $\delta_0 > 0$ depending only on $d, N$ such that if $n$ is a minimizing harmonic map and

$$
\int_{B_1(0)} a_0 |\nabla n|^2 \leq \delta_0,
$$

then $n \in C^\infty(B_{\frac{1}{2}}(0))$. In particular, we have

$$
\int_{B_{\frac{1}{2}}(0)} |\nabla n - \nabla n_{B_{\frac{1}{2}}(0)}|^2 \leq C\theta^2 \int_{B_{\frac{1}{2}}(0)} |\nabla n - \nabla n_{B_{\frac{1}{2}}(0)}|^2.
$$

Note

$$
\int_{B_{\frac{1}{2}}(0)} |\nabla n^{\epsilon_k} - \nabla n^{\epsilon_k}_{B_{\frac{1}{2}}} - \nabla_y \chi \left(\frac{x}{\epsilon}\right) \nabla n^{\epsilon_k}_{B_{\frac{1}{2}}} |^2
\leq \int_{B_{\frac{1}{2}}(0)} |\nabla n^{\epsilon_k} - \nabla n^{\epsilon_k}_{B_{\frac{1}{2}}} - \nabla_y \chi \left(\frac{x}{\epsilon}\right) \nabla n^{\epsilon_k}_{B_{\frac{1}{2}}} |^2 + C \int_{B_{\frac{1}{2}}(0)} |\nabla n - \nabla n_{B_{\frac{1}{2}}(0)}|^2
\leq \int_{B_{\frac{1}{2}}(0)} |\nabla n - \nabla n_{B_{\frac{1}{2}}(0)}|^2 + C \int_{B_{\frac{1}{2}}(0)} |\nabla n - \nabla n_{B_{\frac{1}{2}}(0)}|^2 + O(\epsilon_k)
\leq \theta^2 C \int_{B_{\frac{1}{2}}(0)} |\nabla n - \nabla n_{B_{\frac{1}{2}}(0)}|^2 + O(\epsilon_k).
$$

Therefore if we take $\theta$ small enough such that $C\theta^2 \leq \theta^{2\mu}$, pass to the limit in (3.48), a contradiction follows.

**Lemma 3.23.** Let $\mu, \theta, \epsilon_0, \delta_0$ be as in lemma 3.22. Suppose $n'$ is a minimizer of

$$
\int_{B_{\frac{1}{2}}(0)} a(\bar{x})|\nabla n'|^2.
$$

If

$$
\int_{B_{1}(0)} a\left(\frac{x}{\epsilon}\right) |\nabla n'|^2 \leq \delta_0,
$$

then for all $k$ satisfying $\frac{\epsilon}{\theta^k} \leq \epsilon_0$,

$$
\frac{1}{\theta^{(d-2)k}} \int_{B_{\frac{1}{2}}(0)} a\left(\frac{x}{\epsilon}\right) |\nabla n'|^2 \leq \theta^{2\mu} \int_{B_{1}(0)} a\left(\frac{x}{\epsilon}\right) |\nabla n'|^2 \tag{3.49}
$$

and

$$
\int_{B_{\frac{1}{2}}(0)} |\nabla n' - \nabla y \chi \left(\frac{x}{\epsilon}\right) \nabla n'_{B_{\frac{1}{2}}} - \nabla n'_{B_{\frac{1}{2}}} |^2 dx
\leq \theta^{2\mu} \int_{B_{\frac{1}{2}}(0)} |\nabla n' - \nabla y \chi \left(\frac{x}{\epsilon}\right) \nabla n'_{B_{\frac{1}{2}}} - \nabla n'_{B_{\frac{1}{2}}} |^2 dx + \theta^{2\mu} \frac{1 - \theta^{2\mu}}{1 - \theta \mu} \tag{3.50}
$$

**Proof:** The proof is by induction on $k$. $k = 1$ is exactly the conclusion of previous lemma. Now let $k$ satisfying $\epsilon/\theta^k \leq \epsilon_0$ and suppose (3.49) and (3.50) hold. Define

$$
w_{\epsilon}(z) = n'(\theta^k z). \tag{3.51}
$$
If $n^*$ is a minimizer of $\int_{B_1(0)} a \left( \frac{x}{\epsilon} \right) |\nabla w|^2$ satisfying
$$\int_{B_1(0)} a \left( \frac{\theta^k x}{\epsilon} \right) |\nabla w^*|^2 = \frac{1}{\theta^{d-2k}} \int_{\partial B_{\theta^k}(0)} a \left( \frac{\theta^k x}{\epsilon} \right) |\nabla n^*|^2 \leq \theta^{2k} \mu \int_{B_1(0)} a \left( \frac{x}{\epsilon} \right) |\nabla n^*|^2 \leq \delta_0,$$

Apply Lemma 3.22 to $w^*$, we obtain
$$\frac{1}{\theta^{d-2}} \int_{B_{\theta}(0)} a \left( \frac{\theta^k x}{\epsilon} \right) |\nabla w^*|^2 \leq \theta^2 \mu \int_{B_1(0)} a \left( \frac{\theta^k x}{\epsilon} \right) |\nabla n^*|^2$$

and
$$\int_{B_{\frac{3}{2}}(0)} |\nabla w^* - \nabla \chi \left( \frac{\theta^k x}{\epsilon} \right) \nabla w^* B_{\frac{3}{2}} - \nabla w^* B_{\frac{3}{2}}|^2 \leq \int_{B_{\frac{3}{2}}(0)} |\nabla w^* - \nabla \chi \left( \frac{\theta^k x}{\epsilon} \right) \nabla w^* B_{\frac{3}{2}}|^2 + \theta^2 \mu$$

Rewrite (3.52) and (3.53) utilizing (3.51) and (3.49), we have
$$\frac{1}{\theta^{d-2(k+1)}} \int_{B_{\theta^k}(0)} a \left( \frac{x}{\epsilon} \right) |\nabla n^*|^2 \leq \theta^{2(k+1)} \mu \int_{B_1(0)} a \left( \frac{x}{\epsilon} \right) |\nabla n^*|^2$$

and
$$\int_{B_{\frac{3}{2}}(0)} |\nabla n^* (x) - \nabla \chi \left( \frac{x}{\epsilon} \right) \nabla n^* B_{\frac{3}{2}} - \nabla n^* B_{\frac{3}{2}}|^2 \leq \theta^{2(k+1)} \mu \int_{B_{\frac{3}{2}}(0)} |\nabla n^* - \nabla \chi \left( \frac{x}{\epsilon} \right) \nabla n^* B_{\frac{3}{2}} - \nabla n^* B_{\frac{3}{2}}|^2 + \theta^2 \mu \frac{1 - \theta^{2(k+1)} \mu}{1 - \theta^2 \mu}$$

The next lemma constitutes a priori interior Lipschitz estimate for minimizers of $\int_{\Omega} a \left( \frac{x}{\epsilon} \right) |\nabla n^*|^2$. For simplicity, we state it for minimizers on $B_1(0)$, the most general case will follow by localization and scaling arguments.

**Lemma 3.24.** There exists $\delta_0 > 0$ depending only on $d, N$ satisfies the following: if $n^*$ is a minimizer of $\int_{B_1(0)} a \left( \frac{x}{\epsilon} \right) |\nabla n|^2$ satisfying
$$\int_{B_1(0)} a \left( \frac{x}{\epsilon} \right) |\nabla n^*|^2 dx \leq \delta_0,$$

then there exists a constant $C$ depending only on $d, N, \mu$ such that
$$|\nabla n^*|_{L^\infty(B_{\frac{3}{2}}(0))} \leq C \left( \int_{B_1(0)} |\nabla n^*|^2 \right)^{\frac{1}{2}}$$

**Proof:** We denote by $C$ a generic constant depending on $d, N$ possibly changing from one estimate to another. Let $k$ be such that
$$\epsilon / \theta^k \leq \epsilon_0 < \epsilon / \theta^{k+1}.$$

Substituting this $k$ into Lemma 3.323 we see that
$$\int_{B_{\frac{3}{2}}(0)} |\nabla n^* - \nabla \chi \left( \frac{x}{\epsilon} \right) \nabla n^* B_{\frac{3}{2}} - \nabla n^* B_{\frac{3}{2}}|^2 \leq C \left( \frac{\epsilon}{\epsilon_0} \right)^{2k} \int_{B_{\frac{3}{2}}(0)} |\nabla n^* - \nabla \chi \left( \frac{x}{\epsilon} \right) \nabla n^* B_{\frac{3}{2}} - \nabla n^* B_{\frac{3}{2}}|^2 + C.$$
From which it follows
\[ \int_{B_{\frac{\epsilon}{2}}(0)} |\nabla n^\epsilon - \nabla n^\epsilon_{B_{\frac{\epsilon}{2}}}|^2 \leq C \left( \frac{\epsilon}{\epsilon_0} \right)^2 \int_{B_{\frac{\epsilon}{2}}(0)} |\nabla n^\epsilon - \nabla n^\epsilon_{B_{\frac{\epsilon}{2}}}|^2 + C \left( \int_{B_{\frac{\epsilon}{2}}(0)} \nabla n^\epsilon \right)^2 \]
\[ \leq C_1 \epsilon^{2\mu} + C_2. \quad (3.54) \]

Let \( w^\epsilon(x) = n^\epsilon(\epsilon x) \), then \( w^\epsilon \) is a minimizer of \( \int_{B_{\frac{1}{2}}(0)} a_{\alpha\beta} D\alpha D\beta \). Rescaling (3.54) we have
\[ \int_{B_{\frac{1}{2\epsilon}}(0)} |\nabla w^\epsilon - \nabla w^\epsilon| \leq C_1 \epsilon^{2+2\mu} + C_2 \epsilon^2. \]

From Lipschitz estimates for minimizers of \( \int_{B_{1}(0)} a(x) |\nabla n| \), we know
\[ |\nabla w^\epsilon|_{L^\infty(B_{\frac{1}{2\epsilon}}(0))} \leq C(\int_{B_{\frac{1}{2\epsilon}}(0)} |\nabla w^\epsilon - \nabla w^\epsilon|^2 + (\int_{B_{\frac{1}{2\epsilon}}(0)} \nabla w^\epsilon)^2)^{\frac{1}{2}} \]
\[ \leq C \epsilon^{2+2\mu} + C \epsilon^2 \quad (3.55) \]

Rewrite (3.3) in terms of \( n^\epsilon \), we have
\[ |\nabla n^\epsilon|_{L^\infty(B_{\frac{1}{2\epsilon}}(0))} \leq C \left( \int_{B_{1}(0)} |\nabla n^\epsilon|^2 \right)^{\frac{1}{2}}. \]

REFERENCES


Fanghua Lin, Courant Institute, 251 Mercer Street, New York, NY 10012.
E-mail: linf@cims.nyu.edu

Xiaodong Yan, Courant Institute, 251 Mercer Street, New York, NY 10012.
E-mail: xiayan@cims.nyu.edu

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