DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **9**, Number **1**, January **2003**
Website: http://AIMsciences.org
pp. **1–30**

A TYPE OF HOMOGENIZATION PROBLEM

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1. Introduction.

We consider the following homogenization problem (For relevant discussions, see [BK, L1]). Let Ω be a smooth bounded domain in R^d with a periodic structure, Ω_0 is a periodic subdomain of Ω with $|\Omega_0| = \gamma|\Omega|$ for some given constant $\gamma > 0$. N is a smooth compact submanifold of $\mathbb{R}^k$. We consider

$$\min \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon(x) \cdot D^\beta n^\epsilon(x) dx$$

subject to, with constants $1 \leq \alpha, \beta \leq d$, $c_1 > 0$, $c_1 + c_2 > 0$,

$$\begin{aligned} &a_{\alpha\beta}(x) = \delta_{\alpha\beta}(c_1 + c_2\chi_{\Omega_0})I_k \in M^{k\times k}; \\ &n^\epsilon : \Omega \to N, \qquad\qquad n^\epsilon|_{\partial\Omega} = g. \end{aligned}$$

Here $M^{k\times k}$ being the set of all $k \times k$ matrices, I_k is the identity matrix on $\mathbb{R}^k$.

The question we are concerned is the regularity for n^ϵ and the asymptotic behavior as ϵ tends to zero. The problem can be viewed as an analogue of the usual Γ convergence type problem (see for example, [Ms]) onto curved targets. Due to this constraint in the target, we need to apply techniques used for harmonic maps to construct comparison functions in proving the homogenization limit. We follow the ideas in [AL1] to obtain uniform small energy Hölder estimates and Lipschitz estimates. Such uniform Hölder or $W^{1,p}$ estimates were also found in [C] for some different nonlinear homogenization problems using rather different approaches.

The paper is designed as follows. In section 2, we prove partial regularity result of minimizer n^ϵ for fixed ϵ. We obtain a similar estimates on the size of the singular set as for minimizing harmonic maps. In section 3, we prove the homogenization limit theorem and uniform apriori estimates of n_ϵ independent of ϵ. We also point out an interesting application of our uniform estimates to obtain a uniform bound on the number of singularities of n^ϵ in a special case.

2. Regularity of n^ϵ.

Let Ω be a bounded smooth domain of $\mathbb{R}^d$, $A \subset \Omega$ is a smooth subset of Ω with $|A| = \gamma|\Omega|$, N is a smooth compact submanifold of $\mathbb{R}^k$. We consider the following minimization problem:

$$\min\left\{\int_\Omega a_{\alpha\beta} D^\alpha n \cdot D^\beta n dx, n \in H^1(\Omega, N), n|_{\partial\Omega} = g\right\}, \tag{2.1}$$

where

$$a_{\alpha\beta}(x) = \delta_{\alpha\beta}(1 + \chi_A)I_k \in M^{k\times k}.$$

1991 *Mathematics Subject Classification.* 35,49.
Key words and phrases. Nonlinear homogenization,energy minimizing map,uniform estimates,homogenized problem,singularity.

The existence of a minimizer is standard. For simplicity of notation, we define

$$H^1_g(\Omega, N) = \{n \in H^1(\Omega, \mathbb{R}^k), n(x) \in N \text{ a.e. and } n|_{\partial\Omega} = g\}$$

and we are interested in obtaining some regularity results for the minimizer.

First we derive the Euler-Lagrange equation for a minimizer. Let $N_\epsilon = \{x \in \mathbb{R}^k, dist(x, N) < \epsilon\}$ be a small tubular neighborhood of N on which nearest point projection Π onto N is well defined. Consider $n + s\xi$ where $\xi = (\xi_1, \xi_2, \cdots, \xi_k) \in C_0^\infty(\Omega, \mathbb{R}^k)$. For s small enough, $n + s\xi$ lies in N_ϵ and the following mapping

$$n^s = \Pi \circ (n + s\xi)$$

is an admissible mapping with

$$D^\alpha n^s = D^\alpha n + s(d\Pi_n \circ D^\alpha \xi + \text{Hess}\Pi_n(\xi, D^\alpha n)) + o(s).$$

Therefore

$$\begin{aligned}
0 &= \frac{d}{ds}|_{s=0} E(n^s) = \frac{d}{ds}|_{s=0} \int_\Omega a_{\alpha\beta}(x) D^\alpha n^s(x) \cdot D^\beta n^s(x) dx \\
&= 2\int_\Omega a_{\alpha\beta}(x) D^\alpha n(x) \cdot d\Pi_n(D^\beta \xi(x)) + a_{\alpha\beta}(x) D^\alpha n(x) \cdot \text{Hess}\Pi_n(\xi, D^\beta n) \\
&= 2\int_\Omega a_{\alpha\beta}(x) D^\alpha n \cdot D^\beta \xi - a_{\alpha\beta}(x)(A_n(D^\alpha n, D^\beta n)) \cdot \xi \\
&= 2\int_\Omega \sum_{\alpha=1}^d \{(1+\chi_A) D^\alpha n \cdot D^\alpha \xi - (1+\chi_A) A_n(D^\alpha n, D^\alpha n) \cdot \xi\} = 0, \qquad (2.2)
\end{aligned}$$

here A_n is the second fundamental form of N at $n(x)$.

Partial regularity result for n then follows from a more general theorem:

Theorem 2.1. *[Theorem 1 and 2, [Sh]] Let $\Omega \subset \mathbb{R}^d$ be a smooth open set, $E = \int_\Omega a_{\alpha\beta} D^\alpha n \cdot D^\beta n$, $a_{\alpha\beta}(x) \in L^\infty$ satisfying $\Lambda^{-1} I_d \le a_{\alpha\beta}(x) \le \Lambda I_d$, where Λ is a positive constant, I_d is the $d \times d$ unit matrix. Assume N is a smooth compact Riemannian manifold, n is an E-minimizing map from Ω to N, then there exists a $\epsilon = \epsilon(\Lambda) > 0$ such that if $r^{2-d} \int_{B_r(x)} |\nabla n|^2 \le \epsilon$, then $n \in C^\alpha(B_{\frac{r}{2}}(x))$ for some $0 < \alpha < 1$. Thus n is locally Hölder continuous outside a relatively closed subset $\mathcal{S}_n$ of Ω. Moreover, $H^{d-2}(\mathcal{S}_n) = 0$.*

Meyers' example ([Gi]) show that C^α regularity for general case is optimal. For our case, the coefficient is piecewise constant, we can actually prove the following lipschitz partial regularity result.

Theorem 2.2. *Let $a_{\alpha\beta} = \delta_{\alpha\beta}(c_1 + c_2\chi_A)$, $c_1 > 0$, $c_1 + c_2 > 0$ are given constants. Then any $E-$minimizing map n is locally lipschitz continuous on $\Omega \backslash \mathcal{S}_n$.*

The proof of theorem 2.2 depends on a standard blow up argument and the observation that $\nabla n \in L^p_{loc}$ for some $p > 2$. Our analysis uses strong convergence of the blow up coefficients. We remark that the same arguments therefore is also applicable to the case when $a_{\alpha\beta}$ is piecewise continuous but would fail in general case when $a_{\alpha\beta}$ are merely bounded and measurable.

The lipschitz regularity theorem follows from small energy estimates. An important ingredient in proving small energy estimates is the following monotonicity formula. For simplicity of notation, we shall always assume $a_{\alpha\beta} = \delta_{\alpha\beta}(1 + \chi_A)$.

Denote

$$\begin{aligned}\mathbb{E}(n,r,x) &= \frac{1}{r^{d-2}}\int_{B_r(x)} a_{\alpha\beta}(y)D^\alpha n(y)\cdot D^\beta n(y)dy\\ &= \frac{1}{r^{d-2}}\int_{B_r(x)}(1+\chi_A)|\nabla n(y)|^2dy, \end{aligned} \tag{2.3}$$

we have

Lemma 2.1. *There are constants c and R_0 depending only on d, A such that*

$$\mathbb{E}(n,r,x)\le c\mathbb{E}(n,R,x) \tag{2.4}$$

for any $x\in\Omega$, $\overline{B_R(x)}\subset\Omega$ and all $r\le R\le R_0$.

Proof of Lemma 2.1: Note the lemma is trivial for $d=2$, we assume $d>2$.

Case I: $B_R(x)\subset A$ or $B_R(x)\subset\Omega\setminus A$. We prove the case when $B_R(x)\subset\Omega\setminus A$, the other case is proved in the same way. For $\sigma\in(r,R)$, take comparison map defined by

$$v_\sigma(x)=\begin{cases} n(\frac{\sigma x}{|x|}) & |x|<\sigma,\\ n(x) & |x|\ge\sigma.\end{cases} \tag{2.5}$$

By minimality of n, we have

$$\begin{aligned}&\int_{B_\sigma(x)}|\nabla n|^2=\int_{B_\sigma(x)}(1+\chi_A)|\nabla n|^2\le\int_{B_\sigma(x)}(1+\chi_A)|\nabla v|^2=\int_{B_\sigma(x)}|\nabla v|^2\\ &=(d-2)^{-1}\sigma\left(\int_{\partial B_\sigma(x)}|\nabla n|^2-\int_{\partial B_\sigma(x)}\left|\frac{\partial n}{\partial r}\right|^2\right),\end{aligned}$$

which is

$$0\le\sigma^{2-d}\int_{|x|=\sigma}\left|\frac{\partial n}{\partial r}\right|^2\le\frac{d}{d\sigma}\left(\sigma^{2-d}\int_{B_\sigma(x)}|\nabla n|^2dy\right),\qquad\forall\sigma\in(r,R). \tag{2.6}$$

Integrate (2.6) from r to R, we have

$$r^{2-d}\int_{B_r(x)}|\nabla n|^2dy\le R^{2-d}\int_{B_R(x)}|\nabla n|^2dy.$$

Case II: $x\in\partial A$, there exists a R_0 depending only A such that $\partial A\cap B(x,R_0)$ can be expressed as a graph of a C^2 function for any $x\in\partial A$. Moreover, R_0 can be chosen in such a way that there exists a $\lambda>0, \lambda R_0\le\frac{1}{2}$, for any $w\in H^1(B(x,R_0),\mathbb{R}^k)$ and any $\sigma\le R_0$,

$$(1-\lambda\sigma)\int_{B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla w|^2\le\int_{B_\sigma(x)}(1+\chi_A)|\nabla w|^2\le(1+\lambda\sigma)\int_{B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla w|^2. \tag{2.7}$$

For $R\le R_0$ and any $\sigma\in(r,R)$, let

$$v_\sigma(x)=\begin{cases} n(\frac{\sigma x}{|x|}) & |x|\le\sigma,\\ n(x) & |x|>\sigma.\end{cases} \tag{2.8}$$

By minimality of n and (2.7) we have

$$
\begin{aligned}
&(1-\lambda\sigma)\int_{B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla n|^2 \leq \int_{B_\sigma(x)}(1+\chi_A)|\nabla n|^2 \\
&\leq \int_{B_\sigma(x)}(1+\chi_A)|\nabla v_\sigma|^2 \leq (1+\lambda\sigma)\int_{B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla v_\sigma|^2 \\
&= (1+\lambda\sigma)(d-2)^{-1}\sigma\left[\int_{\partial B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla n|^2 - \int_{\partial B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})\left|\frac{\partial n}{\partial r}\right|^2\right].
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\frac{d}{d\sigma}\left\{\sigma^{2-d}(1+\lambda\sigma)^{2(d-2)}\int_{B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla n|^2\right\} \\
&\geq \sigma^{2-d}(1+\lambda\sigma)^{2(d-2)}\int_{\partial B_\sigma(x)}(1+\chi_{\mathbb{R}^d_+})|\frac{\partial n}{\partial r}|^2 \geq 0.
\end{aligned}
\tag{2.9}
$$

Integrate (2.9) from r to R, we obtain

$$
r^{2-d}(1+\lambda r)^{2(d-2)}\int_{B_r(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla n|^2 \leq R^{2-d}(1+\lambda R)^{2(d-2)}\int_{B_R(x)}(1+\chi_{\mathbb{R}^d_+})|\nabla n|^2.
\tag{2.10}
$$

(2.10) together with (2.7) gives

$$
\begin{aligned}
&r^{2-d}(1+\lambda r)^{2(d-2)-1}\int_{B_r(x)}(1+\chi_A)|\nabla n|^2 \\
&\leq R^{2-d}(1+\lambda R)^{2(d-2)}(1-\lambda R)^{-1}\int_{B_R(x)}(1+\chi_A)|\nabla n|^2.
\end{aligned}
\tag{2.11}
$$

Inequality (2.4) then follows from (2.11) with $c = 2^{2d}$ and R_0 small enough depending only on A.

Case III: $x \notin \partial A$ and $|B_R(x)\cap A| > 0, |B_R(x)\cap A^c| > 0$. $R \leq R_0$, here R_0 is as in case II.

1) $d(x,\partial A) \geq \frac{1}{4}R$,
   a) If $r > \frac{1}{4}R$, then
   $$\mathbb{E}(n,r,x) \leq 4^{d-2}\mathbb{E}(n,R,x).$$
   b) If $r \leq \frac{1}{4}R < d(x,\partial A)$, we can apply case I to $B_r(x) \subset B_{\frac{1}{4}R}(x)$ and obtain
   $$\mathbb{E}(n,r,x) \leq \mathbb{E}(n,\frac{1}{4}R,x) \leq 4^{d-2}\mathbb{E}(n,R,x).$$
2) $d(x,\partial A) < \frac{1}{4}R$,
   a) If $r \geq \frac{1}{4}R$, we still have
   $$\mathbb{E}(n,r,x) \leq 4^{d-2}\mathbb{E}(n,R,x).$$
   b) If $d(x,\partial A) \leq r \leq \frac{1}{4}R$, then we can find $y \in \partial A$ such that $B_r(x) \subset B_{2r}(y) \subset B_{\frac{R}{2}}(y) \subset B_R(x)$. Hence
   $$\mathbb{E}(n,r,x) \leq 2^{d-2}\mathbb{E}(n,2r,y) \leq 2^{d-2}c\mathbb{E}(n,\frac{R}{2},y) \leq 4^{d-2}c\mathbb{E}(n,R,x).$$

c) If $r \leq d(x, \partial A) = l < \frac{1}{4}R$, then we can find $y \in \partial A$ such that $B_r(x) \subset B_l(x) \subset B_{2l}(y) \subset B_{\frac{R}{2}}(y) \subset B_R(x)$, apply case I and case II we have

$$\mathbb{E}(n, r, x) \leq \mathbb{E}(n, l, x) \leq 2^{d-2}\mathbb{E}(n, 2l, y) \leq c\mathbb{E}(n, \frac{R}{2}, y) \leq c\mathbb{E}(n, R, x).$$

The lemma then holds for all $r \leq R \leq R_0$ where c, R_0 depends only on d, A. $\Box$

Remark 2.1. For $r \leq 1, x_0 \in \mathbb{R}^d$, let $A_{x_0,r} = \{x, x_0 + rx \in A\}$. Examine the proof of lemma (2.1) carefully, we see that the same proof shows (2.4) holds for all n_r with constants c, R_0 independent of $r \leq 1, x_0$, here n_r is a minimizer of functional $I_r = \int_\Omega (1 + \chi_{A_{x_0,r}})|\nabla n|^2$. .

If we take the radial derivative term into consideration in the above argument we can prove more. Set $n_{x_0,\lambda}(x) = n(x_0 + \lambda x)$ for $\lambda \in (0, 1]$, $a(x) = 1 + \chi_A$, then

$$\int_{B_1(0)} a(x_0 + \lambda x)|\nabla n_{x_0.\lambda}(x)|^2 dx = \frac{1}{\lambda^{d-2}} \int_{B_\lambda(x_0)} a(x)|\nabla n(x)|^2 dx.$$

Lemma 2.2. *There is a sequence $\lambda_i \to 0$, $\lambda_i \in (0, 1]$ such that n_{x_0,λ_i} converges weakly in $W^{1,2}(B_1(0), N)$ to a limiting map $n_{x_0} \in W^{1,2}(B_1(0), N)$ satisfying $\frac{\partial n_{x_0}}{\partial r} = 0$ a.e. in $B_1(0)$.*

Proof: The proof follows directly from the monotonicity formula and a similar argument as in [SU1].

From lemma 2.1 can also prove the following Cacciopoli type inequality.

Lemma 2.3. *Let n be an energy minimizer of (2.1), Λ be a given constant, if $R^{2-d} \int_{B_R(x_0)} |\nabla n|^2 \leq \Lambda$ for some ball $B_R(x_0)$ with closure contained in Ω, then*

$$\rho^{d-2} \int_{B_{\frac{\rho}{2}}(y)} |\nabla n|^2 \leq C\rho^{-d} \int_{B_\rho(y)} |n - \overline{n}_{y,\rho}|^2$$

for each $y \in B_{\frac{R}{2}}(x_0), \rho < \frac{R}{4}$. Here $C = C(d, N, \Lambda, A) > 0$.

Proof: The proof of lemma 1 in section 2.8 of [Si] can be carried through in our case with only slight changes. We refer the reader to their proof.

A direct result of the Caccioppoli's inequality is the following reverse Hölder inequality. The proof is standard (see e.g. [Gi]).

Lemma 2.4. *Let $a(x) = 1 + \chi_A$. If n is a minimizer of $I = \int_\Omega a(x)|\nabla n|^2 dx$ in $H^1_g(\Omega, N)$, $\overline{B_R(x)} \subset \Omega$ and for some given Λ, we have $R^{2-d} \int_{B_R(x)} a(y)|\nabla n|^2 \leq \Lambda$, then there exists $p > 2$ such that $|Dn| \in L^p_{loc}(\Omega)$ and for $\rho < \frac{R}{4}$, $y \in B_{\frac{R}{2}(x)}$ we have*

$$\left\{ \fint_{B_{\frac{\rho}{2}}(y)} |Dn|^p \right\}^{\frac{1}{p}} \leq C \left\{ \fint_{B_\rho(y)} |Dn|^2 dx \right\}^{\frac{1}{2}},$$

where C, p depend only on d, N, A, Λ.

To show that n is locally Lipschitz continuous on $\Omega \backslash \mathcal{S}_n$. After a suitable translation, rotation and scaling, it reduces to showing the following statement in the normalized situation:

Let $A=\{(x,y)\in B_1^{d-1}(0)\times\mathbb{R} : y>\phi(x)\}$ and ϕ is a $C^{1,\gamma}$ function on $B_1^{d-1}(0)$ with $\phi(0)=|\nabla\phi(0)|=0$ and $||\phi||_{C^{1,\gamma}}\le 1$, then any minimizer n of $\int_{B_1(0)}(1+\chi_A)|\nabla n|^2dx$, u is Lipschitz continuous in $B_{\frac{1}{2}}(0)\subset\mathbb{R}^d$.

We let $\|\phi\|_{C^{1,\gamma}(B_1)}=K(1)$ and define $K(r)=\|\phi^r\|_{C^{1,\gamma}(B_1)}$, for $0<r<1$, where $\phi^r(x)=\frac{1}{r}\phi(rx)$. Thus $K(r)\le r^\gamma K(1)$, $0<r<1$. We then have the following statement.

Lemma 2.5. *Let $a(x)=1+\chi_A$, $\lambda\le 1$. There exists constant $\delta_0,\theta\in(0,1),\mu\in(0,1)$ depending only on d,N such that for any minimizer n_λ of $I_\lambda=\int_{B_1(0)}a(\lambda x)|\nabla n|^2dx$ satisfying*

$$\int_{B_1(0)}a(\lambda y)|\nabla n_\lambda(y)|^2dy\le\delta_0,$$

we have

$$\frac{1}{\theta^{d-2}}\int_{B_\theta(0)}a(\lambda y)|\nabla n_\lambda|^2\le\int_{B_1(0)}a(\lambda x)|\nabla n_\lambda|^2, \tag{2.12}$$

and

$$⨍_{B_{\frac{\theta}{4}}(0)}|D^\lambda n_\lambda-\overline{D^\lambda n_\lambda}_{B_{\frac{\theta}{4}}(0)}|^2\le\theta^{2\mu}⨍_{B_{\frac{1}{4}}(0)}|D^\lambda n_\lambda-\overline{D^\lambda n_\lambda}_{B_{\frac{1}{4}}(0)}|^2, \tag{2.13}$$

here $D^\lambda n_\lambda=\{(1+\chi_{A_\lambda})D_dn_\lambda,D_1n_\lambda,\cdots,D_{d-1}n_\lambda\}$, $D_in=\frac{\partial n}{\partial x_i},i=1,\cdots,d$. $A_\lambda=\{x,\lambda x\in A\}$.

Proof: (2.12) follows from small energy estimates in [Sh] (Proof of theorem 1 in [Sh]). We prove (2.13) by a blow up argument. If (2.13) were not true, there would exist ϵ_k,n_k,λ_k such that n_k is a minimizer of $\int_\Omega(1+\chi_{A_{\lambda_k}})|\nabla n|^2$ with

$$\int_{B_1(0)}(1+\chi_{A_{\lambda_k}})|\nabla n_k|^2=\epsilon_k^2\downarrow 0 \tag{2.14}$$

but

$$⨍_{B_{\frac{\theta}{4}}(0)}|D^{\lambda_k}n_k-\overline{D^{\lambda_k}n_k}_{B_{\frac{\theta}{4}}(0)}|^2>\theta^{2\mu}⨍_{B_{\frac{1}{4}}(0)}|D^{\lambda_k}n_k-\overline{D^{\lambda_k}n_k}_{B_{\frac{1}{4}}(0)}|^2.$$

Let

$$m^k(x)=\frac{n_k(x)-a_k}{\epsilon_k},\qquad a_k=⨍_{B_1(0)}n_kdx. \tag{2.15}$$

Then m^k is a bounded sequence in $H^1(B_1(0),\mathbb{R}^k)$. Passing to a subsequence if necessary, we may assume m^k converges weakly to $m\in H^1(B(0,1),\mathbb{R}^k)$. Since each A_k is a scaling of A with a scaling constant smaller than one and A is a smooth set, the perimeter $P(A_{\lambda_k},B_1(0))$ is finite and we can assume $\chi_{A_{\lambda_k}\cap B_1(0)}\to\chi_{\mathbb{R}^d_+\cap B_1(0)}$. Since n_k is a minimizer of $\int_\Omega(1+\chi_{A_{\lambda_k}})|\nabla n|^2$, we have

$$\int_{B_1(0)}\sum_{\alpha=1}^d(1+\chi_{A_{\lambda_k}})D^\alpha m^k\cdot D^\alpha\eta dx=\epsilon_k\int_{B_1(0)}\sum_{\alpha=1}^d(1+\chi_{A_{\lambda_k}})A_{n_k}(D^\alpha m^k,D^\alpha m^k)\eta dx \tag{2.16}$$

and

$$\int_{B_1(0)}\sum_{\alpha=1}^d(1+\chi_{\mathbb{R}^d_+})D^\alpha m\cdot D^\alpha\eta dx=0 \tag{2.17}$$

for any $\eta \in C_0^\infty(B(0,1),\mathbb{R}^k)$. Subtracting (2.17) from (2.16) we find

$$\int_{B_1(0)} \sum_{\alpha=1}^{d} \Big\{(1+\chi_{A_k})D^\alpha m^k \cdot D^\alpha \eta - (1+\chi_{\mathbb{R}^d_+})D^\alpha m \cdot D^\alpha \eta\Big\}\, dx$$
$$= \epsilon_k \int_{B_1(0)} \sum_{\alpha=1}^{d} (1+\chi_{A_k})A_{n_k}(D^\alpha m^k, D^\alpha m^k)\eta dx. \quad (2.18)$$

By (2.14) and lemma 2.4, we can find some $p>2$ depending only d, N,

$$(\fint_{B(0,\frac{1}{2})} |\nabla n_k|^p dx)^{\frac{1}{p}} \le C(\fint_{B(0,1)} |\nabla n_k|^2 dx)^{\frac{1}{2}} \quad (2.19)$$

for some constant C depending only on d, N.
After rescaling, (2.19) reads

$$(\fint_{B(0,\frac{1}{2})} |\nabla m^k|^p dz)^{\frac{1}{p}} \le C(\fint_{B(0,1)} |\nabla m^k|^2 dz)^{\frac{1}{2}}. \quad (2.20)$$

It follows that $|\nabla m^k|$ is bounded in $L^p(B_{\frac{1}{2}}(0))$. Moreover, a similar argument as in lemma 4.1 of [Ev1], we conclude that m^k is bounded in $L^s(B_{\frac{7}{8}}(0))$ for all $1 \le s < \infty$. Let q satisfy $\frac{2}{p} + \frac{1}{q} = 1$. By approximation the identity (2.18) holds for $\eta \in H_0^1(B_1(0),\mathbb{R}^k) \cap L^q(B_1(0),\mathbb{R}^k)$. We now insert $\eta = \xi^2(m^k - m)$ into (2.18). Here $\xi \equiv 1$ in $B_{\frac{1}{4}}(0)$ and $\xi \equiv 0$ outside $B_{\frac{3}{8}}(0)$.
The left hand side of (2.18) is

$$\begin{aligned} L_k = & \int_{B_1(0)} (1+\chi_{A_k})|\nabla m^k - \nabla m|^2 \xi^2 + \int_{B_1(0)} (1+\chi_{A_k})\nabla m^k \cdot \nabla\xi 2\xi(m^k - m) \\ & - \int_{B_1(0)} (1+\chi_{\mathbb{R}^d_+})\nabla m^k \cdot \nabla \xi 2\xi(m^k-m) + \int_{B_1(0)} (\chi_{A_k} - \chi_{\mathbb{R}^d_+})\nabla m^k \cdot (\nabla m^k - \nabla m)\xi^2 \\ \ge & \int_{B_{\frac{1}{4}}(0)} |\nabla m^k - \nabla m|^2 dx + o(1). \end{aligned} \quad (2.21)$$

The last inequality follows from the fact that $m^k \to m$ strongly in $L^2(B_1(0))$, $\nabla m^k, \nabla m$ are bounded in $L^p(B(0,\frac{1}{2}))$ and $\chi_{A_k} \to \chi_{\mathbb{R}^d_+}$ strongly in $L^q(B_1(0))$.
The right hand side of (2.18) reads

$$\begin{aligned} R_k &= \epsilon_k \int_{B_1(0)} \sum_{B_1(0)} (1+\chi_{A_k})A_{n_k}(D^\alpha m^k, D^\alpha m^k)\xi^2(m^k - m)dx \\ &\le \epsilon_k C \int_{B_1(0)} |\nabla m^k|^2 \xi^2 |m^k - m| \\ &\le \epsilon_k C \left\{\int_{B_1(0)} |\nabla m^k|^p \xi^{\frac{p}{2}}\right\}^{\frac{2}{p}} \left\{\int_{B_1(0)} \xi^q |m^k - m|^q\right\}^{\frac{1}{q}} \\ &\to 0. \end{aligned} \quad (2.22)$$

Combine (2.21) and (2.22) we obtain

$$\nabla m^k \to \nabla m \text{ strongly in } L^2(B_{\frac{1}{4}}(0)). \quad (2.23)$$

Since m is a weak solution of (2.17), we can find some $\alpha \in (0,1)$ such that

$$\fint_{B_{\frac{\theta}{4}}(0)} |D^0 m - \overline{D^0} m_{\frac{\theta}{4}}|^2 \le C\theta^{2\alpha} \fint_{B_{\frac{1}{4}}(0)} |D^0 m - \overline{D^0} m_{\frac{1}{4}}|^2, \tag{2.24}$$

where $D^0 m = ((1+\chi_{\mathbb{R}^d_+})D_d m, D_1 m, \cdots, D_{d-1} m)$. Pick $\mu < \alpha$, choose θ sufficiently small, a contradiction will then arise from the strong convergence of ∇m^k to ∇m in L^2 and strong convergence of χ_{A_k} to $\chi_{\mathbb{R}^d_+}$ in L^q.□

A standard iteration argument then gives Lipschitz regularity for n. Moreover, it gives the following estimates on the gradients.

Lemma 2.6. *Let n be a minimizer of (2.1). There exists $\delta > 0$ such that if $\frac{1}{r^{d-2}} \int_{B_r(x)} |\nabla n|^2 \le \delta$, then n is Lipschitz continuous in $B_{\frac{r}{2}}(x)$ with $|\nabla n|_{L^\infty(B_{\frac{r}{2}}(x))} \le C\left(\frac{1}{r^{d-2}} \int_{B_r(x)} |\nabla n|^2\right)^{\frac{1}{2}}$. Here $C = C(d,N)$.*

Further more, we could reduce the dimension for the singular set of n. First we quote the following lemmas from Simon's lecture notes [Si], which is originally due to Luckhause ([Lu1, Lu2]).

Lemma 2.7 (Corollary 1, [Si], page 27)**.** *Let N be a smooth compact manifold embedded in $\mathbb{R}^p$ and $\Lambda > 0$. There are $\delta_0 = \delta_0(n, N, \Lambda)$ and $C = C(n, N, \Lambda)$ such that the following hold:*

(1) *If we have $\epsilon \in (0,1)$ and if $u \in W^{1,2}(B_\rho(y); N)$ with $\rho^{2-n} \int_{B_\rho(y)} |\nabla u|^2 \le \Lambda$, and $\epsilon^{-2n}\rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 \le \delta_0^2$, then there is $\sigma \in (\frac{3\rho}{4}, \rho)$ such that there is a function $w = w_\epsilon \in W^{1,2}(B_\rho(y); N)$ which agrees with u in a neighborhood of $\partial B_\sigma(y)$ and which satisfies*

$$\sigma^{2-n} \int_{B_\sigma(y)} |Dw|^2 \le \epsilon\rho^{2-n} \int_{B_\rho(y)} |Du|^2 + \epsilon^{-1} C\rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2.$$

(2) *If $\epsilon \in (0, \delta_0]$, and if $u, v \in W^{1,2}(B_{(1+\epsilon)\rho}(y) \backslash B_\rho(y); N)$ satisfy the inequalities $\rho^{2-n} \int_{B_{(1+\epsilon)\rho}(y)\backslash B_\rho(y)} (|Du|^2 + |Dv|^2) \le \Lambda$ and $\epsilon^{-2n}\rho^{-n} \int_{B_{\rho(1+\epsilon)}(y)\backslash B_\rho(y)} |u - v|^2 < \delta_0^2$, then there is $w \in W^{1,2}(B_{\rho(1+\epsilon)}(y)\backslash B_\rho(y); N)$ such that $w = u$ in a neighborhood of $\partial B_\rho(y)$, $w = v$ is a neighborhood of $\partial B_{(1+\epsilon)\rho}(y)$, and*

$$\rho^{2-n} \int_{B_{\rho(1+\epsilon)}(y)\backslash B_\rho(y)} |Dw|^2$$
$$\le C\rho^{2-n} \int_{B_{\rho(1+\epsilon)}(y)\backslash B_\rho(y)} (|Du|^2 + |Dv|^2) + C\epsilon^{-2}\rho^{-n} \int_{B_{\rho(1+\epsilon)}(y)\backslash B_\rho(y)} |u - v|^2.$$

Lemma 2.8. *There exists a sequence $\lambda_i \to 0$ such that the maps n_{a,λ_i} defined by*

$$n_{a,\lambda_i}(x) = n(a + \lambda_i x) \qquad \text{for } x \in B_1(0)$$

converges strongly in $H^1(B_1(0), N)$ to a map $n_a \in H^1(B_1(0), N)$ which is homogeneous of degree 0. Moreover, if $dist(a, \partial A) > 0$, then n_a is a minimizing harmonic map; if $a \in \partial A$, then n_a is a minimizing map of $\int_{B_1(0)} (1 + \chi_{\mathbb{R}^d_+})|\nabla n|^2$.

Proof: The argument in section 2.9 of [Si] can be carried over with only slight modification. We refer the reader to their proof.

Theorem 2.3. *Let $a(x) = (c_1 + c_2\chi_A)$ with $c_1 > 0, c_1 + c_2 > 0$ and A being a smooth subset of Ω. Then the interior singular set $\mathcal{S}_n$ for any minimizer n of $\int_\Omega a(x)|\nabla n|^2$ has Hausdorff dimension less than or equal to $d-3$, in particular, $\mathcal{S}_n$ is a discrete set of points when $d=3$.*

Proof: We can follow essentially the same argument of [SU1] section 5 or Theorem 4.5 of [HL]. We refer readers to their papers. □

Under the additional assumption that $g \in C^{1,\alpha}(\partial\Omega)$, we can have the following.

Theorem 2.4. *Let $g \in C^{1,\alpha}(\partial\Omega, N)$. If n is a minimizer of $\int_\Omega (1+\chi_A)|\nabla n|^2$ in $H^1_g(\Omega, N)$, then the singular set $\mathcal{S}_n$ of n is a compact subset of the interior of Ω; in particular, n is $C^{1,\alpha}$ in a full neighborhood of $\partial\Omega$.*

Proof: Note $A \subset\subset \Omega$, the same argument in [SU2] applies in our case and the boundary regularity of n follows. □

In general case where $a_{\alpha\beta}$ is only bounded and measurable, the monotonicity formula is lacking, we can not carry out the above argument to further reduce the dimension of $\mathcal{S}_n$. Nonetheless, under additional assumptions on N, this can be done. Assume N is a simply connected smooth compact submanifold of $\mathbb{R}^k$, $a_{\alpha\beta}(x)$ are bounded measurable functions. We consider the regularity of a minimizer of $\int_\Omega a_{\alpha\beta}(x)D^\alpha n \cdot D^\beta n$ in $H^1_g(\Omega, N)$.
First we quote the following extension lemma from[HL](a simple version in the case $N = S^2$ can be found in [HKL])

Lemma 2.9 (Theorem 6.2, [HL])**.** *Let N be a simply connected smooth compact submanifold of $\mathbb{R}^k$. If $u \in W^{1,2}(\Omega, N)$ and $a \in \Omega$, then for almost every positive $r < dist(a, \partial\Omega)$, there is a function $w \in W^{1,2}(B_r(a), N)$ such that $w = u$ on $\partial B_r(a)$ and*

$$\int_{B_r(a)} |Dw|^2 \le C\left\{\int_{\partial B_r(a)} |\nabla_{tan} u|^2 \cdot \int_{\partial B_r(a)} |u-\xi|^2 dS\right\}^{\frac{1}{2}},$$

where $\xi \in \mathbb{R}^k$ is arbitrary and C is an absolute constant.

Lemma 2.10. *There exists a positive constant $C = C(d, N)$ such that for any minimizer n of $\int_\Omega a(x)|\nabla w|^2$ in $H^1_g(B_1(0), N)$, we have the following uniform energy bound:*

$$\int_{B_r(0)} |\nabla n|^2 \le \frac{C(d,N)}{1-r} \qquad \textit{for } 0 \le r < 1.$$

Proof: The proof of Theorem 3.1 in [HKL] can be carried over directly to our case. □.

Let $E = \int_{B_R} a^{ij}_{\alpha\beta}(x) D^\alpha v_i D^\beta v_j dx$,
$\mathcal{F} = \{\Sigma, \Sigma \subset B_R$ closed and $\Sigma \subset sing v$ for some minimizer v of $E\}$, then the following hold (for a proof, see e.g. [L2]):

Lemma 2.11. •
a) *If $\Sigma \in \mathcal{F}$, then $\frac{\Sigma - \{x\}}{\lambda} \cap B_R \in \mathcal{F}$ for $|x| < R, 0 < \lambda < R - |x|$.*
b) *$\mathcal{F}$ is compact under the Hausdorff metric.*
c) *$H^{n-2}(\Sigma) = 0$ for all $\Sigma \in \mathcal{F}$.*

Note that a direct result of the lemma is that there exists a $\delta = \delta(N) > 0$ such that $H^{n-2-\delta}(\Sigma) = 0$ for all $\Sigma \in \mathcal{F}$.

3. **Homogenization case.** In this section, we return to the homogenization problem. As in classical theory of homogenization, we are interested in determining the asymptotic behavior of solutions to the above minimization problem. Typically, this analysis amounts to the knowledge of apriori bounds on the norms of the solutions which are valid uniformly in the small parameter ϵ and ensure the compactness of the family $\{n^\epsilon\}_{\epsilon>0}$ in a suitable function space. Before we prove the main results, we first introduce some notations used in this section. We shall always use Einstein's summation principle in this section.

$$\begin{aligned}
&\mathbb{E}_\epsilon(n,r,x) = \frac{1}{r^{n-2}}\int_{B_r(x)} a_{\alpha\beta}\left(\frac{y}{\epsilon}\right) D^\alpha n(y)\cdot D^\beta n(y)dy,\\
&I_\epsilon = a^\epsilon(n,n) = \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n(x)\cdot D^\beta n(x)dx,\\
&Y:\ \text{unit cell in } \mathbb{R}^d, \qquad (f) = \frac{1}{|Y|}\int_Y f(x)dx,\\
&(u,v) = \int_\Omega u\cdot v dx.
\end{aligned}$$

We define

$$W(Y) = \{\phi|\phi\in H^1(Y,\mathbb{R}^k), \phi = (\phi_1,\cdots,\phi_k) \text{ periodic in } Y\}$$

for $\phi,\psi\in H^1(Y,\mathbb{R}^k)$, we set

$$a_1(\phi,\psi) = \int_Y a_{\alpha\beta}(y)D^\alpha\phi(y)\cdot D^\beta\psi(y)dy,$$

and we introduce

$$P_j^\beta(y) = \{\underbrace{0,\cdots,0,}_{j} y^\beta, 0,\cdots,0\}$$

with $\beta\in\mathbb{R}^d, |\beta| = 1, 1\le j\le k$ and define

$$\begin{aligned}
&\chi_j^\beta \in W(Y), \qquad \text{such that}\\
&a_1(\chi_j^\beta - P_j^\beta,\psi) = 0 \qquad \forall\psi\in W(Y).
\end{aligned}\tag{3.1}$$

Since χ_j^β is uniquely defined up to a constant, the following quantity is uniquely defined

$$q_{\alpha\beta}^{ij} = \frac{1}{|Y|}a_1(\chi_j^\beta - P_j^\beta, \chi_i^\alpha - P_i^\alpha)\tag{3.2}$$

and

$$a(u,v) = \int_\Omega q_{\alpha\beta}^{ij}D^\alpha u_i D^\beta v_j dx.$$

In particular, for $a_{\alpha\beta}^{ij}(x) = \delta^{ij}(c_1 + c_2\chi_{\Omega_0})\delta_{\alpha\beta}$, the above equality (3.1) and (3.2) gives

$$q_{\alpha\beta}^{ij} = a_0\delta^{ij}\delta_{\alpha\beta}\tag{3.3}$$

for some constant $a_0>0$ uniquely determined by c_1, c_2 and Ω_0.

Remark 3.2. Note that $q_{\alpha\beta}^{ij}$ can be given an "adjoint" form. We define

$$a_1^*(\phi,\psi) = a_1(\psi,\phi) \qquad \forall\psi,\phi\in H^1(Y,\mathbb{R}^k),$$

and we define $\chi_j^{\beta*}$ by

$$a_1^*(\chi_j^{\beta*} - P_j^\beta,\psi) = 0 \qquad \forall\psi\in W(Y).$$

We then have the formula (See e.g. [BLP])

$$q^{ij}_{\alpha\beta} = \frac{1}{|Y|} a_1^*(\chi_i^{\alpha *} - P_i^\alpha, \chi_j^{\beta *} - P_j^\beta). \tag{3.4}$$

We shall also need some standard results and notations from [BLP]. We denote

$$A^\epsilon = -\frac{\partial}{\partial x^\alpha}\left(a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) \frac{\partial}{\partial x^\beta}\right).$$

We expand $A^\epsilon = \epsilon^{-2}A_1 + \epsilon^{-1}A_2 + \epsilon^0 A_3$, where

$$\begin{aligned} A_1 &= -\frac{\partial}{\partial y^\alpha}\left(a^{ij}_{\alpha\beta}(y)\frac{\partial}{\partial y^\beta}\right), \\ A_2 &= -\frac{\partial}{\partial y^\alpha}\left(a^{ij}_{\alpha\beta}(y)\frac{\partial}{\partial x^\beta}\right) - \frac{\partial}{\partial x^\alpha}\left(a^{ij}_{\alpha\beta}(y)\frac{\partial}{\partial y^\beta}\right), \\ A_3 &= -\frac{\partial}{\partial x^\alpha}\left(a^{ij}_{\alpha\beta}(y)\frac{\partial}{\partial x^\beta}\right). \end{aligned}$$

A^* denotes the adjoint operator of A.

3.1. **Homogenization limit.** In this section, we prove the following theorem about the homogenization limit.

Theorem 3.5. *For any sequence $\{n^\epsilon\}$, where n^ϵ is a minimizer I_ϵ, there exists a subsequence n^{ϵ_k} such that n^{ϵ_k} converges weakly to a minimizing harmonic map n in $H^1_g(\Omega, N)$. Moreover, there exists some constant $a_0 > 0$ uniquely determined by $a^{\alpha\beta}$ such that*

$$\lim_{\epsilon\to 0}\int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon dx \to a_0 \int_\Omega |\nabla n|^2.$$

We shall prove the theorem in two steps. First we show that n is a weakly harmonic map (lemma 3.12), we then show that n is a minimizing harmonic map and the energy convergence results (lemma 3.15).

Lemma 3.12. *For any sequence of $\{n^\epsilon\}$, n^ϵ being a minimizer of I_ϵ in $H^1_g(\Omega, N)$, there exists a subsequence n^{ϵ_k} such that n^{ϵ_k} converges weakly in $H^1_g(\Omega, N)$ to a weakly harmonic map n.*

Proof: Let ϵ_l be a subsequence such that

$$\int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon_l}\right) D^\alpha n^{\epsilon_l} \cdot D^\beta n^{\epsilon_l} \to \liminf_{\epsilon\to 0}\int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon.$$

By assumption we have

$$\int_\Omega |\nabla n^{\epsilon_l}|^2 \le \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon_l}\right) D^\alpha n^{\epsilon_l} \cdot D^\beta n^{\epsilon_l} dx \le 2\int_\Omega |\nabla n^1|^2 dx \le C.$$

Therefore n^{ϵ_l} is a bounded sequence in $H^1_g(\Omega, \mathbb{R}^k)$, hence a subsequence (we still denote by n^{ϵ_l}) converges weakly in $H^1_g(\Omega, \mathbb{R}^k)$, strongly in $L^2(\Omega, \mathbb{R}^k)$ and pointwise almost everywhere to $n \in H^1_g(\Omega, \mathbb{R}^k)$. Since n^{ϵ_l} is a minimizer of $\int_\Omega a_{\alpha\beta}(\frac{x}{\epsilon_l}) D^\alpha n \cdot D^\beta n dx$ in $H^1_g(\Omega, N)$, n^{ϵ_l} is a weak solution of the following Euler-Lagrange equation:

$$-D^\beta\left(a_{\alpha\beta}\left(\frac{x}{\epsilon_l}\right) D^\alpha n^{\epsilon_l}(x)\right) = a_{\alpha\beta}\left(\frac{x}{\epsilon_l}\right)(A_{n^{\epsilon_l}}(D^\alpha n^{\epsilon_l}, D^\beta n^{\epsilon_l})).$$

To illustrate the main idea, from now on, we assume $N = S^n$, the general target case could be proved similarly (though technically more complicated). In this case, n^{ϵ_l} is a weak solution of

$$-\mathrm{div}((c_1 + c_2\chi_{\Omega_{\epsilon_l}})\nabla n_j^{\epsilon_l}) = (c_1 + c_2\chi_{\Omega_{\epsilon_l}})|\nabla n^{\epsilon_l}|^2 n_j^{\epsilon_l}, \quad 1 \le j \le k. \tag{3.5}$$

Here $\Omega_\epsilon = \{x, \frac{x}{\epsilon} \in \Omega_0\}$.
Following the idea of [Ev1], we write equation (3.5) in the form

$$\begin{aligned}-\mathrm{div}((c_1 + c_2\chi_{\Omega_{\epsilon_l}})\nabla n_j^{\epsilon_l}) &= (c_1 + c_2\chi_{\Omega_{\epsilon_l}})|\nabla n^{\epsilon_l}|^2 n_j^{\epsilon_l} \\ &= \sum_{\alpha=1}^{d}\sum_{q=1}^{k}(c_1 + c_2\chi_{\Omega_{\epsilon_l}})\left[\frac{\partial n_q^{\epsilon_l}}{\partial x^\alpha}\left\{\frac{\partial n_q^{\epsilon_l}}{\partial x^\alpha} n_j^{\epsilon_l} - \frac{\partial n_j^{\epsilon_l}}{\partial x^\alpha} n_q^{\epsilon_l}\right\}\right].\end{aligned}$$

Let

$$b_{\epsilon,\alpha}^{qj} = (c_1 + c_2\chi_{\Omega_\epsilon})\left\{\frac{\partial n_q^\epsilon}{\partial x^\alpha} n_j^\epsilon - \frac{\partial n_j^\epsilon}{\partial x^\alpha} n_q^\epsilon\right\}, \tag{3.6}$$

then from the following lemma 3.13, one concludes that $b_\epsilon^{qj} = \{b_{\epsilon,\alpha}^{qj}\}$ satisfies $\mathrm{div}(b_\epsilon^{qj}) = 0$ weakly for each $1 \le q, j \le k$. Denote $a_{\alpha\beta}(x) = (a_{\alpha\beta}^{ij}(x))$, set

$$\xi_{\epsilon,\beta}^j = a_{\alpha\beta}^{ij}\left(\frac{x}{\epsilon}\right) D^\alpha n_i^\epsilon,$$

we see $\xi_{\epsilon_l,\beta}^j$ is bounded in $L^2(\Omega)$. Therefore we can extract a subsequence, we still denote by $\xi_{\epsilon_l,\beta}^j$ for simplicity of notation, such that

$$\xi_{\epsilon_l,\beta}^j \rightharpoonup \xi_\beta^j \text{ weakly in } L^2(\Omega).$$

Taking into account that n^ϵ is bounded in L^∞ and converge strongly in L^2 to n, we obtain

$$b_{\epsilon_l,\alpha}^{qj} \rightharpoonup \xi_\alpha^q n_j - \xi_\alpha^j n_q \qquad \text{weakly in } L^2.$$

For each q, j, apply the Div-Curl lemma (see e.g. [Ev2] or [Mu]) to $\sum_{\alpha=1}^{d} \frac{\partial n_q^{\epsilon_l}}{\partial x^\alpha} b_{\epsilon_l,\alpha}^{qj}$ we obtain

$$\frac{\partial n_q^{\epsilon_l}}{\partial x^\alpha} b_{\epsilon,\alpha}^{qj} \rightharpoonup \frac{\partial n_q}{\partial x^\alpha}(\xi_\alpha^q n_j - \xi_\alpha^j n_q) \text{ in } \mathcal{D}'(\Omega).$$

Therefore the limit equation for ξ_β^j is

$$\int_\Omega \xi_\beta^j \frac{\partial \phi}{\partial x^\beta} dx = \sum_{\alpha=1}^{d}\sum_{q=1}^{k}\int_\Omega \frac{\partial n_q}{\partial x^\alpha}(\xi_\alpha^q n_j - \xi_\alpha^j n_q)\phi dx, \qquad \forall \phi \in C_0^\infty(\Omega). \tag{3.7}$$

We compute ξ_β^j using adjoint functions. We introduce

$$P = \{P_j(y)\}_{j=1}^k, \qquad P_j(y) = \text{homogeneous polynomial of degree 1},$$

and we define w such that

$$\begin{aligned} &A_1^* w = 0 \text{ in } Y, \\ &w - P \in W(Y). \end{aligned} \tag{3.8}$$

If we set

$$w - P = -\chi \tag{3.9}$$

then the equation (3.8) is equivalent to

$$a_1^*(\chi - P, \psi) = 0 \qquad \forall \psi \in W(Y).$$

We then introduce

$$w_\epsilon(x) = \{\epsilon w_j\left(\frac{x}{\epsilon}\right)\}.$$

We observe that

$$A^{\epsilon *} w_\epsilon = 0, \tag{3.10}$$

and that

$$w_\epsilon(x) = P(x) - \{\epsilon \chi_j\left(\frac{x}{\epsilon}\right)\}.$$

For $\phi \in C_0^\infty(\Omega)$, we set

$$\phi n = \{\phi n_1, \cdots, \phi n_k\}.$$

Choose

$$v = \phi w_\epsilon$$

as a test function in (3.7), and multiply (3.10) by ϕn^ϵ, we obtain

$$\int_\Omega \xi^j_{\epsilon,\beta}(D^\beta(\phi w_{\epsilon j}) - \phi D^\beta w_{\epsilon j})dx - \int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\beta w_{\epsilon j}(D^\alpha(\phi n^\epsilon_i) - \phi D^\alpha n^\epsilon_i)dx$$
$$= \int_\Omega D^\alpha n^\epsilon_q b^{qj}_{\epsilon,\alpha}\phi w_{\epsilon j}dx. \tag{3.11}$$

But one verifies that

$$D^\beta(\phi w_{\epsilon j}) - \phi D^\beta w_{\epsilon j} \to D^\beta(\phi P_j) - \phi D^\beta P_j \qquad \text{strongly in } L^2(\Omega),$$
$$D^\alpha(\phi n^\epsilon_j) - \phi D^\alpha n^\epsilon_j \to D^\alpha(\phi n_j) - \phi D^\alpha n_j \text{ strongly in } L^2.$$

and that

$$\begin{aligned}\int_\Omega b^{qj}_{\epsilon,\alpha} D^\alpha n^\epsilon_q \phi w_{\epsilon j} \qquad &= \int_\Omega D^\alpha n^\epsilon_q b^{qj}_{\epsilon,\alpha}\phi P_j - \epsilon\int_\Omega D^\alpha n^\epsilon_q b^{qj}_{\epsilon,\alpha}\phi\chi_j\left(\frac{x}{\epsilon}\right)dx \\ &\to \int_\Omega D^\alpha n_q(\xi^q_\alpha n_j - \xi^j_\alpha n_q)\phi P_j.\end{aligned} \tag{3.12}$$

The last part of (3.12) follows from the fact that

$$D^\alpha n^\epsilon_q b^{qj}_{\epsilon,\alpha} \rightharpoonup D^\alpha n_q(\xi^q_\alpha n_j - \xi^j_\alpha n_q) \text{ in } \mathcal{D}'(\Omega)$$

and that

$$\int_\Omega |D^\alpha n^\epsilon_q b^{qj}_{\epsilon,\alpha}\phi\chi_j\left(\frac{x}{\epsilon}\right)|dx \le C(N)|\chi|_{L^\infty}\|\nabla n^\epsilon\|_{L^2}.$$

On the other hand, as $\epsilon \to 0$,

$$a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\beta_y w_{\epsilon j}\left(\frac{x}{\epsilon}\right) = \left(a^{ij}_{\alpha\beta}D^\beta_y(w_j)\right)\left(\frac{x}{\epsilon}\right) \to (a^{ij}_{\alpha\beta}D^\beta_y w_j)$$

in L^∞ weak star, so that passing to the limit in (3.11) gives

$$\int_\Omega \xi^j_\beta(D^\beta(\phi P_j) - \phi D^\beta P_j)dx - (a^{ij}_{\alpha\beta}D^\beta_y w_j)\int_\Omega(D^\alpha(\phi n_i) - \phi D^\alpha n_i)dx$$
$$= \int_\Omega D^\alpha n_q(\xi^q_\alpha n_j - \xi^j_\alpha n_q)\phi P_j dx \tag{3.13}$$

But $\int_\Omega D^\alpha(\phi n_i)dx = 0$ and the right hand side of (3.13) equals $\int_\Omega \xi^j_\beta D^\beta(\phi P_j)dx$, therefore (3.13) reduces to

$$-\int_\Omega \xi^j_\beta D^\beta P_j\phi + (a^{ij}_{\alpha\beta}D^\beta_y w_j)\int_\Omega \phi D^\alpha n_i dx = 0,$$

i.e.

$$\xi^j_\beta D^\beta P_j = (a^{ij}_{\alpha\gamma} D^\gamma_y w_k) D^\alpha n_i. \tag{3.14}$$

We now take $P = P^\beta_j$, then $w = P^\beta_j - \chi^{\beta *}_j$ and (3.14) gives

$$\begin{aligned}\xi^j_\beta &= (a^{ik}_{\alpha\gamma} D^\gamma (P^\beta_{jk} - \chi^{\beta *}_{jk})) D^\alpha n_i \\ &= \frac{1}{|Y|} a^*_1 (\chi^{\beta *}_j - P^\beta_j, -P^\alpha_i) D^\alpha n_i \\ &= q^{ij}_{\alpha\beta} D^\alpha n_i,\end{aligned}$$

so that (using (3.2), (3.3) and (3.4))

$$(\xi^j_\beta, D^\beta v_j) = a(n, v) \qquad \forall v \in W^{1,2}_0(\Omega, \mathbb{R}^k),$$

and n is therefore a weak solution of

$$-\text{div}(a_0 \nabla n) = a_0 |\nabla n|^2 n.$$

Remark 3.3. For general compact manifold N, we can basically follow the same idea used above to show that the weak limit n is a weakly harmonic map. But we have to adapt to the work of [Be] to choose appropriate orthonormal frame on $T_{n(x)}N$ to rewrite the equation (3.5) into a similar form as (3.8). We then can prove the homogenization limit n is a weakly harmonic map.

Lemma 3.13. *For each $\phi \in C^\infty_0(\Omega)$, $b^{qj}_{\epsilon,\alpha}$ defined by (3.6), we have*

$$\int_\Omega b^{qj}_{\epsilon,\alpha} D^\alpha \phi(x) dx = 0$$

for all $1 \le q, j \le k$.

Proof: We compute

$$\begin{aligned}\int_\Omega b^{qj}_{\epsilon,\alpha} D^\alpha \phi(x) \quad &= \sum_{\alpha=1}^d \int_\Omega \frac{\partial \phi}{\partial x_\alpha} (c_1 + c_2 \chi_{\Omega_\epsilon}) \left(\frac{\partial n^\epsilon_q}{\partial x_\alpha} n^\epsilon_j - \frac{\partial n^\epsilon_j}{\partial x_\alpha} n^\epsilon_q \right) dx \\ &= \sum_{\alpha=1}^d \int_\Omega (c_1 + c_2 \chi_{\Omega_\epsilon}) \frac{\partial n^\epsilon_q}{\partial x^\alpha} \frac{\partial (\phi n^\epsilon_j)}{\partial x^\alpha} - \int_\Omega (c_1 + c_2 \chi_{\Omega_\epsilon}) \frac{\partial n^\epsilon_j}{\partial x^\alpha} \frac{\partial (\phi n^\epsilon_q)}{\partial x^\alpha} \\ &= \int_\Omega (c_1 + c_2 \chi_{\Omega_\epsilon}) |\nabla n^\epsilon|^2 n^\epsilon_q n^\epsilon_j \phi - \int_\Omega (c_1 + c_2 \chi_{\Omega_\epsilon}) |\nabla n^\epsilon|^2 n^\epsilon_j n^\epsilon_q \phi dx \\ &= 0.\end{aligned}$$

Lemma 3.14. *Let $\chi^\alpha_k = \{\chi^\alpha_{kj}\}$ be given by (3.1). If n^ϵ is a minimizer of I_ϵ and $n^\epsilon \rightharpoonup n$ weakly in $W^{1,2}(\Omega, N)$, then $\forall F(x) = (F^l_\gamma(x)) \in W^{1,2} \cap L^\infty(\Omega, M^{k \times d})$,*

$$\lim_{\epsilon \to 0} \int_\Omega a^{ij}_{\alpha\beta} \left(\frac{x}{\epsilon} \right) D^\alpha_y \chi^\gamma_{il} \left(\frac{x}{\epsilon} \right) F^l_\gamma(x) D^\beta n^\epsilon_j(x) dx = 0.$$

Here $M^{k \times k}$ being the set of all $k \times k$ matrices.

Proof: Let $\phi \in C_0^\infty(\Omega)$. Since n^ϵ is a weak solution of the Euler -Lagrange equation (2.2), we have

$$\begin{aligned}
&\int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\beta n^\epsilon_j(x) D^\alpha_y \chi^\gamma_{il}\left(\frac{x}{\epsilon}\right) F^l_\gamma(x)\phi(x)dx\\
&= \int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\beta n^\epsilon_j(x)\epsilon\left(D^\alpha_x\left(\chi^\gamma_{il}\left(\frac{x}{\epsilon}\right) F^l_\gamma(x)\phi(x)\right) - \chi^\gamma_{il}\left(\frac{x}{\epsilon}\right) D^\alpha_x(F^l_\gamma(x)\phi(x))\right)dx\\
&= \epsilon\int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) A_{n^\epsilon}(D^\alpha n^\epsilon, D^\beta n^\epsilon)\cdot\chi^\gamma_l\left(\frac{x}{\epsilon}\right) F^l_\gamma(x)\phi(x)dx\\
&\quad - \epsilon\int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\beta n^\epsilon_j(x)\chi^\gamma_{il}\left(\frac{x}{\epsilon}\right) D^\alpha(F^l_\gamma(x)\phi(x))dx\\
&\le \epsilon C|\chi|_{L^\infty}(\|Dn^\epsilon\|^2_{L^2}|F\phi|_{L^\infty} + \|Dn^\epsilon\|_{L^2}\|D(F\phi)\|_{L^2})\\
&\to 0.
\end{aligned}$$

Since ϕ is arbitrary, we conclude the lemma.

Lemma 3.15. *Let n be as in lemma 3.12, then n is a minimizing harmonic map in $H^1_g(\Omega, N)$ and*

$$\int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x)dx \to a_0\int_\Omega |\nabla n|^2 dx.$$

Proof: To show that n is actually a minimizing harmonic map subject to its boundary constraints, we need to introduce the correctors. Let m_ϵ be a cut-off function defined as follows

$$\begin{aligned}
&m_\epsilon \in \mathcal{D}(\Omega),\\
&m_\epsilon(x) = 0 \qquad \text{if } d(x,\partial\Omega) \le \epsilon,\\
&m_\epsilon(x) = 1 \qquad \text{if } d(x,\partial\Omega) \ge 2\epsilon,\\
&\epsilon^{|\gamma|}|D^\gamma m_\epsilon(x)| \le c_\gamma, \forall\gamma\in\mathbb{N}.
\end{aligned}$$

Here c_γ depends on γ but does not depend on ϵ.
For fixed positive number L, we define ${}^L\eta^j_\beta \in C_0^\infty(M^{k\times d}, M^{k\times d})$ by

$$ {}^L\eta^j_\beta(y) = \begin{cases} y^j_\beta & |y| \le L\\ \textit{smooth} & L < |y| < L+1\\ 0 & |y| \ge L+1 \end{cases} \tag{3.15}$$

with

$$\int_{L<|y|<L+1} ({}^L\eta^j_\beta(y))^2 dy < \frac{1}{L^2}.$$

We consider

$$\mu^L_\epsilon(n) = \{-\epsilon m_\epsilon(x)\chi^p_{\beta i}(\frac{x}{\epsilon}){}^L\eta^\beta_p(Dn(x))\}^k_{i=1},$$

where $\chi^p_\beta = \{\chi^{pi}_\beta\}$ is defined by (3.1). Let $w \in H^1_g(\Omega, N)$ be a given function, when ϵ is small enough, $w + \mu^L_\epsilon(w)$ lies in a small neighborhood of N on which the nearest point projection Π is well defined, then

$$w^L_\epsilon = \Pi\circ(w + \mu^L_\epsilon(w)) \in H^1_g(\Omega, N)$$

and we have

$$X^L_\epsilon(w) = a_1(w^L_\epsilon, w^L_\epsilon) = \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha w^L_\epsilon(x)\cdot D^\beta w^L_\epsilon(x)dx,$$

where

$$D^\alpha w^L_\epsilon(x) = D^\alpha w(x) - m_\epsilon(x) D^\alpha_y \chi^\beta_k\left(\frac{x}{\epsilon}\right)^L \eta^k_\beta(Dw(x)) - r^\alpha_\epsilon(x)$$

where

$$\begin{aligned} r^\alpha_\epsilon(x) = \epsilon \Big\{ & d\Pi_n \circ D^\alpha_x \left(m_\epsilon(x)\chi^\beta_p(\frac{x}{\epsilon})^L\eta^p_\beta(Dw(x))\right) \\ & + \mathrm{Hess}\Pi_\mathrm{n}\left(\mathrm{m}_\epsilon(\mathrm{x})\chi^\beta_\mathrm{p}\left(\frac{\mathrm{x}}{\epsilon}\right)^\mathrm{L}\eta^\mathrm{p}_\beta(\mathrm{Dw}(\mathrm{x})), \mathrm{D}^\alpha\mathrm{w}\right)\Big\} \\ & - m_\epsilon(x) D^\alpha_y\chi^\beta_p\left(\frac{x}{\epsilon}\right)^L\eta^p_\beta(Dw(x)) + o(\epsilon). \end{aligned}$$

By virtue of the construction of m_ϵ and properties of χ^β_p, ${}^L\eta^j_\beta$, we have

$$r^\alpha_\epsilon \to 0 \text{ in } L^2.$$

Therefore, if we set

$$A^\alpha(w) = D^\alpha w(x) - m_\epsilon(x) D^\alpha_y \chi^\beta_k\left(\frac{x}{\epsilon}\right) {}^L\eta^k_\beta(Dw(x)),$$

and let

$$Y^L_\epsilon(w) = \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) A^\alpha(w)\cdot A^\beta(w), \tag{3.16}$$

we have as $\epsilon \to 0$

$$X^L_\epsilon(w) - Y^L_\epsilon(w) \to 0 \qquad \forall w \in H^1_g(\Omega, N).$$

But we can pass to the limit in (3.16); we obtain (here and in the following we always write $a_{\alpha\beta}(x) = (a^{ij}_{\alpha\beta}(x)) \in M^{k\times k}$)

$$\begin{aligned} \lim_{\epsilon\to 0} Y^L_\epsilon = & \int_\Omega (a^{ij}_{\alpha\beta}) D^\alpha w_i D^\beta w_j - \int_\Omega (a^{ij}_{\alpha\beta} D^\beta_y \chi^\delta_{lj}) {}^L\eta^l_\delta(Dw(x)) D^\alpha w_i dx \\ & - \int_\Omega (a^{ij}_{\alpha\beta} D^\alpha_y \chi^\gamma_{pi}) {}^L\eta^l_\gamma(Dw(x)) D^\beta w_j dx \\ & + \int_\Omega (a^{ij}_{\alpha\beta} D^\beta_y \chi^\delta_{lj} D^\alpha_y \chi^\gamma_{pi}) {}^L\eta^p_\gamma(Dw(x)) {}^L\eta^l_\delta(Dw(x)) dx. \end{aligned} \tag{3.17}$$

We then let $L \to \infty$ in (3.17), by choice of ${}^L\eta^\beta_j$, we have

$$\lim_{L\to\infty}\lim_{\epsilon\to 0} Y^L_\epsilon(w) = \int_\Omega p^{ij}_{\alpha\beta} D^\alpha w_i D^\beta w_j dx \tag{3.18}$$

where

$$p^{ij}_{\alpha\beta} = (a^{ij}_{\alpha\beta}) - (a^{kj}_{\gamma\beta} D^\gamma_y \chi^\alpha_{ik}) - (a^{il}_{\alpha\delta} D^\delta_y \chi^\beta_{jl}) + (a^{kl}_{\gamma\delta} D^\gamma \chi^\alpha_{ik} D^\delta_y \chi^\beta_{jl}).$$

Note

$$p^{ij}_{\alpha\beta} = a_1(P^\alpha_i, P^\beta_j - \chi^\beta_j) = a_1(\chi^\beta_j - P^\beta_j, -P^\alpha_i),$$

(3.18) then gives

$$\lim_{L\to\infty}\lim_{\epsilon\to 0} Y^L_\epsilon(w) = a(w,w). \tag{3.19}$$

From the assumption that n^ϵ is a minimizer of

$$a^\epsilon(n,n) = \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n(x)\cdot D^\beta n(x) dx$$

in $H^1_g(\Omega, N)$, we know for any L fixed,

$$\int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon(x)\cdot D^\beta n^\epsilon(x)dx \le \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha w^L_\epsilon(x)\cdot D^\beta w^L_\epsilon(x)dx = X^L_\epsilon(w),$$

passing to the limit,

$$\limsup_{\epsilon\to 0}\int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x)dx \le \lim_{\epsilon\to 0} X^L_\epsilon(w) = \lim_{\epsilon\to 0} Y^L_\epsilon(w)$$

Let $L\to\infty$, using (3.19), we have

$$\limsup_{\epsilon\to 0}\int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x)dx \le a(w,w). \tag{3.20}$$

On the other hand, we let $z_\epsilon = n^\epsilon - w^L_\epsilon$, we have

$$\begin{aligned}0 &\le a^\epsilon(z^\epsilon, z^\epsilon)\\ &= a^\epsilon(n^\epsilon - w^L_\epsilon, n^\epsilon - w^L_\epsilon)\\ &= a^\epsilon(n^\epsilon, n^\epsilon) - 2a^\epsilon(n^\epsilon, w^L_\epsilon) + a^\epsilon(w^L_\epsilon, w^L_\epsilon).\end{aligned} \tag{3.21}$$

While from lemma 3.12 and lemma 3.14, we have

$$\begin{aligned}a^\epsilon(n^\epsilon, w^L_\epsilon) &= \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\beta n^\epsilon(x)\cdot\Big(D^\alpha w(x) - m_\epsilon(x) D^\alpha_y \chi^\gamma_p(\frac{x}{\epsilon})^L \eta^p_\gamma(Dw(x)) - r^\alpha_\epsilon\Big)dx\\ &= a^\epsilon(n^\epsilon, w) - \int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) m_\epsilon(x) D^\beta n^\epsilon_j(x) D^\alpha_y \chi^\gamma_{pi}\left(\frac{x}{\epsilon}\right)^L \eta^p_\gamma(Dw(x))dx\\ &\quad - \int_\Omega a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\beta n^\epsilon_j(x) r^\alpha_{\epsilon i} dx\\ &\to a(n,w).\end{aligned} \tag{3.22}$$

Plug in $w = n$ to (3.22), together with (3.21) we have

$$0 \le \liminf_{\epsilon\to 0} a^\epsilon(n^\epsilon, n^\epsilon) - a(n,n). \tag{3.23}$$

We then proved

$$\lim_{\epsilon\to 0} a^\epsilon(n^\epsilon, n^\epsilon) = a(n,n).$$

Finally it follows from (3.20), (3.23) and (3.3) that n is a minimizing harmonic map in $H^1_g(\Omega, N)$.

In fact, we could prove the following local convergence lemma:

Lemma 3.16. *Let n^ϵ be as in lemma 3.12, then there exists a subsequence n^{ϵ_k} and a minimizing harmonic map $n \in H^1_g(\Omega, N)$ such that for any $\overline{B_r(x)} \subset \Omega$, we have*

$$\int_{B_r(x)} a_{\alpha\beta}\left(\frac{y}{\epsilon_k}\right) D^\alpha n^{\epsilon_k}(y)\cdot D^\beta n^{\epsilon_k}(y)dy \to \int_{B_r(x)} a_0|\nabla n(y)|^2 dy.$$

Proof: Since n^ϵ is bounded in $H^1_g(\Omega, N)$, we can find a subsequence n^{ϵ_l} and weakly harmonic map n such that $n^{\epsilon_l} \rightharpoonup n \in H^1_g(\Omega, N)$. Let m^r_ϵ be a cut off function defined as follows

$$\begin{aligned}&m^r_\epsilon \in \mathcal{D}(B_r(x)),\\ &m^r_\epsilon(y) = 0 \qquad \text{if } d(y, \partial B_r(x)) \le \epsilon,\\ &m^r_\epsilon(y) = 1 \qquad \text{if } d(y, \partial B_r(x)) \ge 2\epsilon,\\ &\epsilon^{|\gamma|}|D^\gamma m^r_\epsilon(y)| \le c_\gamma, \forall\gamma\in\mathbb{N},\ c_\gamma \text{ depends on } r,\gamma \text{ but not on } \epsilon.\end{aligned} \tag{3.24}$$

For L fixed positive number, ${}^L\eta^j_\beta \in C^\infty(M^{k\times d}, M^{k\times d})$ is defined by (3.15). For any $v \in W^{1,2}(B_r(x), N)$, we consider

$$\mu^L_{\epsilon,r}(v) = \{-\epsilon m^r_\epsilon(x)\chi^\beta_{ki}(\frac{x}{\epsilon}){}^L\eta^k_\beta(Dv(x))\}.$$

For ϵ small enough, we can define

$$v^L_{\epsilon,r} = \Pi \circ (v + \mu^L_{\epsilon,r}(v)). \tag{3.25}$$

Now follow the same proof as in lemma 3.15, we can prove that

$$\lim_{L\to\infty}\lim_{\epsilon\to 0}\int_{B_r(x)} a_{\alpha\beta}\left(\frac{y}{\epsilon}\right) D^\alpha v^L_{\epsilon,r}(y)\cdot D^\beta v^L_{\epsilon,r}(y)dy \to \int_{B_r(x)} a_0|\nabla v|^2 dy. \tag{3.26}$$

Let $a^\epsilon_r(n,n) = \int_{B_r(x)} a_{\alpha\beta}(\frac{y}{\epsilon})D^\alpha n(y)\cdot D^\beta n(y)dy$, $a_r(u,v) = \int_{B_r(x)} a_0 \nabla u\cdot\nabla v$. Take subsequence $n^{\epsilon_{l_k}}$ of $\{n^{\epsilon_l}\}$ such that

$$\begin{aligned}
&a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon_{l_k}}\right) D^\alpha n^{\epsilon_{l_k}}_i(x) \rightharpoonup \xi^j_\beta \text{ weakly in } L^2(B_r(x), M^{k\times d}),\\
&n^{\epsilon_{l_k}} \rightharpoonup n \text{ weakly in } W^{1,2}(B_r(x), N),\\
&a^{\epsilon_{l_k}}_r(n^{\epsilon_{l_k}}, n^{\epsilon_{l_k}}) \to \liminf_{l\to\infty} a^{\epsilon_l}_r(n^{\epsilon_l}, n^{\epsilon_l}).
\end{aligned}\tag{3.27}$$

A similar argument as in lemma 3.12, we can show that

$$(\xi^j_\beta, D^\beta v_j)_r = \int_{B_r(x)} \xi^j_\beta D^\beta v_j = a_r(u,v). \tag{3.28}$$

Using (3.27) and (3.28), we can argue in the same way as in lemma 3.12 and lemma 3.14 to obtain

$$\lim_{L\to\infty}\lim_{l\to\infty}\int_{B_r(x)} a^{ij}_{\alpha\beta}\left(\frac{y}{\epsilon_l}\right) D^\alpha n^{\epsilon_l}_i(y) D^\beta v^L_{\epsilon_l j,r}(y)dy \to \int_{B_r(x)} a_0\nabla n\cdot\nabla v dy. \tag{3.29}$$

On the other hand, we have

$$\begin{aligned}
0 &\le a^{\epsilon_{l_k}}_r(n^{\epsilon_{l_k}} - n^L_{\epsilon_{l_k}}, n^{\epsilon_{l_k}} - n^L_{\epsilon_{l_k}})\\
&= a^{\epsilon_{l_k}}_r(n^{\epsilon_{l_k}}, n^{\epsilon_{l_k}}) - 2a^{\epsilon_{l_k}}_r(n^{\epsilon_{l_k}}, n^L_{\epsilon_{l_k}}) + a^{\epsilon_{l_k}}_r(n^L_{\epsilon_{l_k}}, n^L_{\epsilon_{l_k}}).
\end{aligned}\tag{3.30}$$

By (3.26), (3.27) and (3.29), this implies

$$\liminf_{l\to\infty} a^{\epsilon_l}_r(n^{\epsilon_l}, n^{\epsilon_l}) \ge a_r(n,n).$$

To prove

$$\liminf_{l\to\infty} a^{\epsilon_l}_r(n^{\epsilon_l}, n^{\epsilon_l}) \le a_r(n,n), \tag{3.31}$$

we need to modify the argument in lemma 3.15. Since now n^ϵ does not have the same boundary condition on $\partial B_r(x)$, we need to apply Luckhause's lemma (2.7) to construct suitable comparison functions. Let $\overline{B_{r_0}}(x) \subset \Omega$ and let $\theta \in (0,1)$, $\delta > 0$ be given. Choose any $M \in \mathbb{N}$ with $\limsup_{l\to\infty} \mathbb{E}_{\epsilon_l}(n^{\epsilon_l}, r_0, x) < M\delta$ and note that if $\varepsilon \in (0, 1-\theta/M)$ we must have some integer $l \in \{2,\cdots,M\}$ such that

$${r_0}^{2-d}\int_{B_{r_0(\theta+l\varepsilon)}(x)\setminus B_{r_0(\theta+(l-2)\varepsilon)}(x)} a^{ij}_{\alpha\beta}\left(\frac{y}{\epsilon_k}\right) D^\alpha n^{\epsilon_k}_i(y) D^\beta n^{\epsilon_k}_j(y)dy < \delta$$

for infinitely many ϵ_{l_k}, because otherwise we get that $\mathbb{E}_{\epsilon_l}(n^{\epsilon_l}, r_0, x) > M\delta$ for all sufficiently large l by summation over l, contrary to the definition of M. Thus choose such an l, letting $r = r_0(\theta + (l-2)\varepsilon)$ and noting that $r(1+\varepsilon) \le r_0(\theta + l\varepsilon) < r_0, r \in (\theta r_0, r_0)$ such that

$$r_0{}^{2-d} \int_{B_{r(1+\varepsilon)}(x)\backslash B_r(x)} a^{ij}_{\alpha\beta}\left(\frac{y}{\epsilon_{l_k}}\right) D^\alpha n_i^{\epsilon_{l_k}}(y) D^\beta n_j^{\epsilon_k}(y) dy \le \delta$$

for some subsequence $n^{\epsilon_{l_k}}$ (for simplicity of notation, we shall denote the subsequence by n^{ϵ_k} from now on). Passing to the limit, we then have

$$r_0^{2-n} \int_{B_{r(1+\varepsilon)}(x)\backslash B_r(x)} |\nabla n|^2 dx \le \delta.$$

By lemma 2.7, we can find $w^{\epsilon_k} \in W^{1,2}(B_{r(1+\varepsilon)}(y)\backslash B_\rho(y); N)$ such that $w^{\epsilon_k} = n$ in a neighborhood of $\partial B_r(x)$, $w^{\epsilon_k} = n^{\epsilon_k}$ in a neighborhood of $\partial B_{r(1+\varepsilon)}(x)$ and

$$\begin{aligned} &r^{2-d} \int_{B_{r(1+\varepsilon)}(x)\backslash B_r(x)} |\nabla w^{\epsilon_k}|^2 dx \\ &\le C r^{2-d} \int_{B_{r(1+\varepsilon)}(x)\backslash B_r(x)} (|\nabla n|^2 + |\nabla n^{\epsilon_k}|^2 + \epsilon_k^{-2} r^{-2} |n - n^{\epsilon_k}|^2) dx, \end{aligned}$$

where C depends only on d, N. Now consider $n^L_{\epsilon_k, r}(x)$ defined by formula (3.25), i.e.

$$n^L_{\epsilon,r} = \Pi \circ (n + \mu^L_{\epsilon,r}(n))$$

and let

$$\tilde{n}^{\epsilon_k} = \begin{cases} n^{\epsilon_k} & B_{r_0}(y)\backslash B_{r(1+\varepsilon)}(y) \\ w^{\epsilon_k} & B_{r(1+\varepsilon)}(x)\backslash B_r(x) \\ n^L_{\epsilon_k,r} & B_r(x) \end{cases}$$

Then by minimality of n^{ϵ_k} we have

$$\begin{aligned} &\int_{B_{r(1+\varepsilon)}(x)} a_{\alpha\beta}\left(\frac{y}{\epsilon_k}\right) D^\alpha n^{\epsilon_k}(y) \cdot D^\beta n^{\epsilon_k}(y) dy \\ \le &\int_{B_{r(1+\varepsilon)}(x)} a_{\alpha\beta}\left(\frac{y}{\epsilon_k}\right) D^\alpha \tilde{n}^{\epsilon_k}(y) \cdot D^\beta \tilde{n}^{\epsilon_k}(y) \\ \le &\int_{B_r(x)} a_{\alpha\beta}\left(\frac{y}{\epsilon_k}\right) D^\alpha n^L_{\epsilon_k,r}(y) D^\beta n^L_{\epsilon_k,r}(y) dy + 2\int_{B_{r(1+\varepsilon)}(x)\backslash B_r(x)} |\nabla w^{\epsilon_k}|^2 dx. \end{aligned} \tag{3.32}$$

By (3.26), taking limit in (3.32) gives

$$\liminf_{l\to\infty} \int_{B_r(x)} a_{\alpha\beta}\left(\frac{x}{\epsilon_l}\right) D^\alpha n^{\epsilon_l}(x) \cdot D^\beta n^{\epsilon_l}(x) dx \le \int_{B_r(x)} a_0 |\nabla n|^2 dx + C\delta.$$

Since δ is arbitrary, (3.31) follows. Thus we can find a subsequence such that

$$a_r^{\epsilon_l}(n^{\epsilon_l}, n^{\epsilon_l}) \to a_r(n,n). \square$$

In fact, the above argument actually proves the following statement:

Lemma 3.17. *Assume sequence $n^{\epsilon_j} \rightharpoonup n$ in $H^1_g(\Omega, \mathbb{R}^k)$, where n^ϵ is a minimizer of I_ϵ in $H^1_g(\Omega, N)$. Then for $\overline{B_r(x)} \subset \Omega$, we have*

$$\int_{B_r(x)} a_{\alpha\beta}\left(\frac{y}{\epsilon_j}\right) D^\alpha n^{\epsilon_j}(y) \cdot D^\beta n^{\epsilon_j}(y) dy \to a_0 \int_{B_r(x)} |\nabla n|^2 dy$$

and we can find a subsequence $n^{\epsilon_{j_k}}$ such that

$$\int_{B_r(x)} |D^\alpha n_i^{\epsilon_{j_k}} - D^\alpha n_i - D_y^\alpha \chi_{pi}^\beta \left(\frac{x}{\epsilon_{j_k}}\right) \frac{\partial n_p^{\epsilon_{j_k}}}{\partial x^\beta}|^2 \to 0. \tag{3.33}$$

Proof: Let $n^{\epsilon_{j_k}}$ be such that

$$\int_{B_r(x)} a_{\alpha\beta}\left(\frac{x}{\epsilon_{j_k}}\right) D^\alpha n^{\epsilon_{j_k}} \cdot D^\beta n^{\epsilon_{j_k}} \to \liminf_{j\to\infty} \int_{B_r(x)} a_{\alpha\beta}\left(\frac{x}{\epsilon_j}\right) D^\alpha n^{\epsilon_j} \cdot D^\beta n^{\epsilon_j}.$$

The previous lemma showed that

$$\int_{B_r(x)} a_{\alpha\beta}\left(\frac{x}{\epsilon_{j_k}}\right) D^\alpha n^{\epsilon_{j_k}} \cdot D^\beta n^{\epsilon_{j_k}} \to \int_{B_r(x)} a_0 |\nabla n|^2.$$

Moreover, for each $L > 0$ fixed, we have

$$\begin{aligned} &\int_{B_r(x)} \left| D^\alpha n_i^{\epsilon_{j_k}} - D^\alpha n_i - m_\epsilon^r(x) D_y^\alpha \chi_{ki}^\beta \left(\frac{x}{\epsilon_{j_k}}\right)^L \eta_\beta^k (Dn(x)) \right|^2 \\ \leq &\int_{B_r(x)} a_{\alpha\beta}\left(\frac{x}{\epsilon_{j_k}}\right) \left(D^\alpha n^{\epsilon_{j_k}} - m_{\epsilon_{j_k}}^r(x) D_y^\alpha \chi_k^\gamma \left(\frac{x}{\epsilon_{j_k}}\right)^L \eta_\gamma^k (Dn(x)) \right) \\ &\cdot \left(D^\beta n^{\epsilon_{j_k}} - m_{\epsilon_{j_k}}^r(x) D_y^\beta \chi_k^\delta \left(\frac{x}{\epsilon_{j_k}}\right)^L \eta_\delta^k (Dn(x)) \right). \end{aligned} \tag{3.34}$$

Let $\epsilon_{j_k} \to 0$, then $L \to \infty$, the right hand side of (3.34) converges to $\liminf_{j\to\infty} a_r^{\epsilon_j}(n^{\epsilon_j}, n^{\epsilon_j}) - a_r(n,n) = 0$, (3.33) follows.

3.2. **Hölder estimate.** In this and next section, we prove some uniform small energy estimates on n^ϵ. More precisely, we have

Theorem 3.6. *There exists a constant δ_0 independent of ϵ such that for any $\overline{B_r(x)} \in \Omega$, and any minimizer n^ϵ of I_ϵ satisfying*

$$\mathbb{E}_\epsilon(n^\epsilon, r, x) = \frac{1}{r^{n-2}} \int_{B_r(x)} a_{\alpha\beta}^{ij}(\frac{x}{\epsilon}) D^\alpha n_i^\epsilon D^\beta n_j^\epsilon dx \leq \delta_0$$

then $n^\epsilon \in C^\beta(B_{\frac{r}{2}}(x))$ for all $\beta < 1$.

We prove the theorem following the compactness argument developed by Avellenda and Lin for linear elliptic system (See [AL1, AL2, AL3]) . Namely, we prove the uniform Hölder estimate in three steps.

Step 1. Show that there exist constants $\theta \in (0,1), \mu \in (0,1), \epsilon_0, \delta_0$ depend only on d, N such that if

$$\mathbb{E}_\epsilon(n^\epsilon, 1, 0) \leq \delta_0,$$

then for $\epsilon \leq \epsilon_0$,

$$\mathbb{E}_\epsilon(n^\epsilon, \theta, 0) \leq \theta^{2\mu} \mathbb{E}_\epsilon(n^\epsilon, 1, 0)$$

This step follows directly from the small energy estimates for minimizing harmonic maps and the strong convergence results of $\mathbb{E}_\epsilon(n^\epsilon, r, 0)$ by lemma 3.17.

Step 2. A recursive argument of the step 1 implies

$$\mathbb{E}_\epsilon(n^\epsilon, r, 0) \leq r^{2\mu} \mathbb{E}_\epsilon(n^\epsilon, 1, 0)$$

for all $r \geq \frac{\epsilon}{\epsilon_0}$.

Step 3. Blow up argument in ϵ scale.

Before we present the lemmas, we specify that from now on, by a minimizer of $I(n) = \int_B a_{\alpha\beta} D^\alpha n \cdot D^\beta n$ we mean $I(n) \le I(m)$ for any $m \in H^1(B, N)$ with $m - n$ compactly supported in B.

Lemma 3.18. *For any $0 < \mu < 1$, there exist $\theta, 0 < \theta < 1$, and $\epsilon_0, \delta_0 > 0$ depending only on d and N, such that if n^ϵ is a minimizer of $I_\epsilon = \int_{B_2(0)} a^{ij}_{\alpha\beta}(\frac{x}{\epsilon}) D^\alpha n_i(x) D^\beta n_j(x) dx$ with*

$$\int_{B_1(0)} a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x) dx \le \delta_0,$$

then for all $\epsilon \le \epsilon_0$,

$$\frac{1}{\theta^{d-2}} \int_{B_\theta(0)} a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x) dx \le \theta^{2\mu} \int_{B_1(0)} a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon_i(x) D^\beta n^\epsilon_j(x) dx. \tag{3.35}$$

Proof: Suppose $\mu < \mu' < 1$. Were (3.35) false, then for any fixed $\theta \in (0, 1), \delta > 0$ which will be chosen later, we could find minimizers n^{ϵ_k} of I_{ϵ_k} satisfying

$$\int_{B_1(0)} a^{ij}_{\alpha\beta}(\frac{x}{\epsilon_k}) D^\alpha n^{\epsilon_k}_i D^\beta n^{\epsilon_k}_j \le \delta,$$

yet

$$\frac{1}{\theta^{d-2}} \int_{B_\theta(0)} a^{ij}_{\alpha\beta}(\frac{x}{\epsilon_k}) D^\alpha n^{\epsilon_k}_i(x) D^\beta n^{\epsilon_k}_j(x) dx > \theta^{2\mu} \int_{B_1(0)} a^{ij}_{\alpha\beta}(\frac{x}{\epsilon_k}) D^\alpha n^{\epsilon_k}_i(x) D^\beta n^{\epsilon_k}_j(x) dx. \tag{3.36}$$

By the homogenization limit lemmas 3.12 and 3.15, we know there exists a subsequence (for simplicity, we denote by n^k) such that n^k is a minimizer of I_{ϵ_k} and $n^k \rightharpoonup n$ where n is a minimizing harmonic map and

$$\int_{B_\theta(0)} a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon_k}\right) D^\alpha n^k_i(x) D^\beta n^k_j(x) dx \to a_0 \int_{B_\theta(0} |\nabla n|^2 dx,$$

$$\int_{B_1(0)} a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon_k}\right) D^\alpha n^k_i(x) D^\beta n^k_j(x) dx \to a_0 \int_{B_1(0)} |\nabla n|^2 dx,$$

Since n is a minimizing harmonic map, there exists a constant $\delta_0 > 0$, such that if

$$\int_{B_1(0)} |\nabla n|^2 \le \delta_0$$

then for θ small enough, the following holds

$$\frac{1}{\theta^{d-2}} \int_{B_\theta(0)} |\nabla n|^2 dx \le \theta^{2\mu'} \int_{B_1(0)} |\nabla n|^2 dx.$$

Now take $\delta = \frac{\delta_0}{2}$, pass to the limit in (3.36), a contradiction arises.

Lemma 3.19. *Given $\mu, 0 < \mu < 1$, let $\theta, \epsilon_0, \delta_0$ be as in lemma 3.18. Then for all n^ϵ, n^ϵ being a minimizer of I_ϵ, satisfying*

$$\mathbb{E}_\epsilon(n^\epsilon, 1, 0) \le \delta_0,$$

for all k such that $\epsilon/\theta^k \le \epsilon_0$, we have

$$\mathbb{E}_\epsilon(n^\epsilon, \theta^k, 0) \le \theta^{2k\mu} \mathbb{E}_\epsilon(n^\epsilon, 1, 0). \tag{3.37}$$

Proof: The proof is by induction on k. $k=1$ is exactly the conclusion of lemma 3.18. Now let k satisfying $\epsilon/\theta^k \le \epsilon_0$ and suppose (3.37) holds. Define

$$w^\epsilon(z) = n^\epsilon(\theta^k z).$$

Then $w^\epsilon \in H^1(B_1(0), N)$ and from (3.37)

$$\begin{aligned}\int_{B_1(0)} a_{\alpha\beta}\left(\frac{\theta^k z}{\epsilon}\right) D^\alpha w^\epsilon(z)\cdot D^\beta w^\epsilon(z)dz &= \frac{1}{\theta^{(d-2)k}}\int_{B_{\theta^k}(0)} a_{\alpha\beta}\left(\frac{y}{\epsilon}\right) D^\alpha n^\epsilon(y)\cdot D^\beta n^\epsilon(y)dy \\ &\le \delta_0.\end{aligned} \tag{3.38}$$

and w^ϵ is a minimizer of $\int_{B_2(0)} a_{\alpha\beta}\left(\frac{\theta^k z}{\epsilon}\right) D^\alpha n(z)\cdot D^\beta n(z)dz$. Apply lemma 3.18 to w^ϵ, we obtain

$$\frac{1}{\theta^{d-2}}\int_{B_\theta(0)} a^{ij}_{\alpha\beta}\left(\frac{\theta^k z}{\epsilon}\right) D^\alpha w_i^\epsilon(z) D^\beta w_j^\epsilon(z)dz \le \theta^{2\mu}\int_{B_1(0)} a^{ij}_{\alpha\beta}\left(\frac{\theta^k z}{\epsilon}\right) D^\alpha w_i^\epsilon(z) D^\beta w_j^\epsilon(z)dz. \tag{3.39}$$

Rewriting (3.39) using (3.38) we see that

$$\begin{aligned}&\frac{1}{\theta^{(d-2)(k+1)}}\int_{B_{\theta^{k+1}}(0)} a^{ij}_{\alpha\beta}\left(\frac{y}{\epsilon}\right) D^\alpha n_i^\epsilon(y) D^\beta n_j^\epsilon(ydy) \\ &\le \theta^{2(k+1)\mu}\int_{B_1(0)} a^{ij}_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n_i^\epsilon(x) D^\beta n_j^\epsilon(x)dx.\end{aligned}$$

Remark 3.4. Note that to repeat the above recursive argument for any fixed ball $B(x,r)\subset\Omega$, we actually need modify lemma 3.18 into following version:

Lemma 3.20. *Suppose n^ϵ is a minimizer of $\int_{B_2(0)} a_{\alpha\beta}(\frac{x+x_0}{\epsilon})D^\alpha n\cdot D^\beta n$, x_0 is a fixed point in $\mathbb{R}^d$. Then we can find δ_0 independent of x_0, n^ϵ, such that if*

$$\int_{B_1(0)} a_{\alpha\beta}\left(\frac{x+x_0}{\epsilon}\right) D^\alpha n^\epsilon\cdot D^\beta n^\epsilon \le \delta_0,$$

then

$$\frac{1}{\theta^{d-2}}\int_{B_\theta(0)} a_{\alpha\beta}\left(\frac{x+x_0}{\epsilon}\right) D^\alpha n^\epsilon\cdot D^\beta n^\epsilon \le \theta^{2\mu}\int_{B_1(0)} a_{\alpha\beta}\left(\frac{x+x_0}{\epsilon}\right) D^\alpha n^\epsilon\cdot D^\beta n^\epsilon.$$

Proof: The proof amounts to a strong convergence of the corresponding energy independent of base point x_0. For this purpose, we need only to modify the correctors by the same translation. i.e. we choose correctors by $m^r_\epsilon(x+x_0)D_y\chi(\frac{x+x_0}{\epsilon})\nabla n(x)$, then we obtain the same energy convergence results. The rest is similar. The same argument applies to the recursive argument for Lipschitz estimate in the next section.

The next lemma constitutes a priori interior Hölder estimate for minimizers of $\int a_{\alpha\beta}(\frac{x}{\epsilon})D^\alpha n\cdot D^\beta n$. For simplicity, we state it for minimizers on $B_1(0)$, the most general case will follow by localization and scaling arguments.

Lemma 3.21. *Given $\mu, 0<\mu<1$, there exists $\delta_0>0$ depending only on d, N, such that if n^ϵ is a minimizer of $\int_{B_2(0)} a^{ij}_{\alpha\beta}(\frac{x}{\epsilon})D^\alpha n_i D^\beta n_j$ satisfying*

$$\int_{B_1(0)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon\cdot D^\beta n^\epsilon \le \delta_0,$$

then there exists a constant C depending only on d, N, μ such that

$$[n^\epsilon]_{C^{0,\mu}(B_{\frac{1}{2}}(0))} \le C \int_{B_1(0)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon(x) \cdot D^\beta n^\epsilon(x) dx.$$

Proof: We denote by C a generic constant depending on d, N, μ possibly changing from one estimate to another. From lemma 3.19, we conclude that for all $r \ge \epsilon/\epsilon_0$,

$$\frac{1}{r^{d-2}} \int_{B_r(0)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon(x) \cdot D^\beta n^\epsilon(x) \le C r^{2\mu} \int_{B_1(0)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon(x) \cdot D^\beta n^\epsilon(x). \tag{3.40}$$

We take $r = 2\epsilon/\epsilon_0$ in (3.40) and define the new function

$$w^\epsilon(x) = n^\epsilon(\epsilon x) \qquad x \in B_{\frac{2}{\epsilon_0}}(0). \tag{3.41}$$

Then w^ϵ is a minimizer of $I_1 = \int_{B_{\frac{2}{\epsilon_0}}(0)} a_{\alpha\beta}(x) D^\alpha n \cdot D^\beta n$. From the small energy estimates in section 2, we conclude that there exists a $\delta_1 > 0$, such that if

$$\int_{B_{\frac{2}{\epsilon_0}}(0)} a_{\alpha\beta}(x) D^\alpha w^\epsilon \cdot D^\beta w^\epsilon \le \delta_1,$$

then

$$\sup_{|x|<\frac{1}{\epsilon_0}} \sup_{0<r<\frac{1}{\epsilon_0}} \frac{1}{r^{n-2+2\mu}} \int_{B_r(x)} a_{\alpha\beta}(x) D^\alpha w^\epsilon(x) \cdot D^\beta w^\epsilon(x)$$
$$\le C \frac{1}{\epsilon_0^{d-2}} \int_{B_{\frac{2}{\epsilon_0}}(0)} a_{\alpha\beta}(x) D^\alpha w^\epsilon(x) \cdot D^\beta w^\epsilon(x) dx \tag{3.42}$$

Setting $s = r\epsilon$, plug (3.2) into (3.42) we see that

$$\sup_{|x|<\frac{\epsilon}{\epsilon_0}} \sup_{0<s<\frac{\epsilon}{\epsilon_0}} \frac{1}{s^{n-2+2\mu}} \int_{B_s(x)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon$$
$$\le C \int_{B_{\frac{\epsilon}{\epsilon_0}}(0)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon(x) \cdot D^\beta n^\epsilon(x). \tag{3.43}$$

If we combine (3.40) and (3.43) and small energy estimates from theorem 2.1, the conclusion follows for all ϵ.

Remark 3.5. It can be checked that when $a_{\alpha\beta}$ is bounded measurable, we still have the strong convergence of energy and the homogenization limit is a minimizing harmonic map. We thus conclude that the above uniform Hölder estimates holds for general case.

In fact, if we have the monotonicity formula or assume N is simply connected, we can prove the following interesting lemma from the uniform Hölder estimates.

Corollary 3.1. *(Singular points converge to singular points). Suppose n^ϵ is a sequence of minimizers of $I_\epsilon = \int_\Omega a_{\alpha\beta} D^\alpha n \cdot D^\beta n$ in $H^1_g(\Omega, N)$ converges weakly to n in $H^1_g(\Omega, N)$. Assume N is simply connected or monotonicity formula holds for n^ϵ with a uniform constant, then*

(1) *If y^ϵ is a singular point for n^ϵ such that $y^\epsilon \to y \in \Omega$, then y is a singular point for n.*

(2) *If $y \in \Omega$ is a singular point for n, then for all sufficiently small ϵ, n^ϵ has a singular point at some y^ϵ with $y^\epsilon \to y$.*

Proof: Consider (1). By previous results, we know n is a minimizing harmonic map in $H^1_g(\Omega, n)$. If y is not a singular point of n, then for $r > 0$ small enough, we have

$$\frac{1}{r^{d-2}} \int_{B_r(y)} a_0 |\nabla n|^2 \leq \frac{\delta_0}{2^d},$$

here δ_0 is given by lemma 3.21. By energy convergence result, we know for ϵ small enough,

$$\frac{1}{r^{d-2}} \int_{B_r(y)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \frac{\delta_0}{2^{d-2}}.$$

On the other hand, for ϵ small enough, we have $y^\epsilon \in B_{\frac{r}{4}}(y)$, hence

$$\frac{1}{(\frac{r}{2})^{d-2}} \int_{B_{\frac{r}{2}}(y^\epsilon)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq 2^{d-2} \frac{1}{r^{d-2}} \int_{B_r(y)} a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n^\epsilon \cdot D^\beta n^\epsilon \leq \delta_0,$$

by lemma 3.21, y^ϵ is a regular point of n^ϵ, a contradiction.
With regard to (2). If the conclusion were false, we could find a r and a subsequence n^{ϵ_k} such that there are no singular points of n^{ϵ_k} in $B_r(y)$. Without loss of generality, we may assume $y = 0$. From Lipschitz estimates lemma 2.6 plus the assumption that monotonicity formula holds or N is simply connected, one obtains a uniform bound on $|\nabla n^{\epsilon_k}|_{L^\infty(B_{\frac{r}{2}}(0))}$. For any $\delta > 0$, when r small enough, we have $\frac{1}{r^{d-2}} \int_{B_r(0)} a_{\alpha\beta}\left(\frac{x}{\epsilon_k}\right) D^\alpha n^{\epsilon_k} D^\beta n^{\epsilon_k} \leq \delta$. From strong convergence of the energy, we conclude $\frac{1}{r^{d-2}} \int_{B_r(0)} a_0 |\nabla n|^2 \leq \delta$. When δ is small enough, this implies 0 is a regular point of n. A contradiction. □

In fact, if we assume $a_{\alpha\beta}(x)$ to be continuous, we can consider the homogenization problem

$$\min \int_\Omega a_{\alpha\beta}\left(\frac{x}{\epsilon}\right) D^\alpha n \cdot D^\beta n \tag{3.44}$$

and study the asymptotic behavior of minimizer n^ϵ as ϵ approaches zero. One can check easily that the theorem 3.6 continues to hold in this case. Moreover, n^ϵ converges weakly to a minimizing harmonic map $n \in H^1(\Omega, N)$. Recall from Schoen and Uhlenbeck's result (see [SU1]) the singular set of n^ϵ is of dimension $d-3$, in particular, when $d = 3$, the singular set of n^ϵ is discrete. Later Almgren and Lieb ([AL]) obtained a uniform bound (depends only on geometry of Ω and the energy of the boundary function of n) for the number of singular points of a minimizing harmonic map n from $\Omega \subset \mathbb{R}^3$ onto S^2. In particular, they have the following theorem on uniform distance between singular points:

Theorem 3.7. *[*[AL]*] Theorem 2.1] Suppose n is a minimizing harmonic map from $\Omega \subset \mathbb{R}^3$ into S^2 having a singularity at $y \in \Omega$. Let D denotes the distance from y to $\partial\Omega$. Then there is a universal constant C independent of Ω, n, D, y etc such that there is no other singularity within distance CD of y.*

As an application of our uniform small energy estimates, we obtain the following theorem on the uniform bound for the number of singular points of n^ϵ.

Theorem 3.8. *Let $a_{\alpha\beta}(x)$ be continuous, we consider the homogenization (3.44) for $N = S^2$ when $d = 3$. Then the total number of singular points N_ϵ of n^ϵ is bounded above by some l independent of ϵ.*

Proof: Choose a subsequence n^{ϵ_k} which converges weakly to $n \in H^1_g(\Omega, S^2)$, then n is a minimizing harmonic map and the singular points of n^{ϵ_k} converges to singular point of n. Note singular points of n are isolated and the total number of singular points is bounded above by a constant depending only on d, g and the geometry of Ω. Let p be a singular point of n, then there exists r depending only on d, g, Ω such that there is no other singular point of n in $B_r(p)$. Without loss of generality, we assume $p = 0$. Since singular points of n are limits of n^{ϵ_k}, we can always find singular points p_k of n^{ϵ_k} such that $p_k \to 0$. We claim there exists a $L > 0$ such that for k large enough, all singular points of n^{ϵ_k} close to 0 lie in $B(0, L\epsilon_k)$. Otherwise for a subsequence (we still denote by n^{ϵ_k}), we can always find a singular point p_k of n^{ϵ_k} with $|p_k| = \delta_k \to 0$ and $\frac{\epsilon_k}{\delta_k} \to 0$. Choose $\delta > 0$, we consider $w^{\epsilon_k} = n^{\epsilon_k}(\frac{\delta_k x}{\delta})$. Then w^{ϵ_k} is a minimizer of $\int_{B_2(0)} a_{\alpha\beta}(\frac{\delta_k x}{\epsilon_k \delta}) D^\alpha n \cdot D^\beta n$ with

$$\int_{B_1(0)} |\nabla w^{\epsilon_k}|^2 \leq C.$$

The bound follows from the energy bound in lemma 2.10.
Since $\frac{\epsilon_k \delta}{\delta_k} \to 0$, we can argue as before and show that up to a subsequence w^{ϵ_k} converges weakly to a minimizing harmonic map w and

$$\int_{B_r(0)} a_{\alpha\beta}\left(\frac{\delta_k x}{\delta \epsilon_k}\right) D^\alpha w^{\epsilon_k} \cdot D^\beta w^{\epsilon_k} \to \int_{B_r(0)} a_0 |\nabla w|^2 \tag{3.45}$$

for any $r \leq \frac{1}{2}$. Note each w^{ϵ_k} has a singular point q_k on $\partial B_\delta(0)$. By corollary 3.1, we know q_k converges to a singular point q of w. On the other hand, we note 0 is also a singular point of w. In fact, if 0 is a regular point of w, then for some r small enough, we have

$$r^{2-d} \int_{B_r(0)} a_0 |\nabla w|^2 \leq \frac{1}{2}\delta_0.$$

Here δ_0 is a small constant as in lemma 3.21. By strong convergence of energy (3.45), we conclude that for k large enough, we have

$$r^{2-d} \int_{B_r(0)} a\left(\frac{\delta_k x}{\delta \epsilon_k}\right) |\nabla w^{\epsilon_k}|^2 \leq \delta_0,$$

which implies that

$$\left(\frac{r\delta_k}{\delta}\right)^{2-d} \int_{B_{\frac{r\delta_k}{\delta}}(0)} a\left(\frac{x}{\epsilon_k}\right) |\nabla n^{\epsilon_k}|^2 \leq \delta_0.$$

By uniform energy estimates lemma 3.21 and corollary 3.1, we conclude 0 is a regular point of n, a contradiction to our choice of 0. Therefore w has a singular point at 0 and $\partial B_\delta(0)$. Since w is a minimizing harmonic map from $B_1(0)$ into S^2, for any singular point p of w lies in $B_{\frac{1}{2}}(0)$, we conclude from theorem 3.7 there exists a r independent of w, such that there are no other singular points of w in $B_r(p)$. If we take δ small enough, that would be a contradiction. Therefore there exists a L such that all singular points of n^{ϵ_k} close to 0 lies in $B_{L\epsilon_k}(0)$ for k large enough.
The conclusion of the theorem now follows easily. In fact, modify the proof of theorem 2.1 in [AL] slightly, one can show that at ϵ scale the distance between the singular points of n^ϵ is given by $C\epsilon$ with C independent of n^ϵ. Hence there are M singular points of n^ϵ in $B_{L\epsilon}(p)$ for each singular point p of n, with M independent

of ϵ. Since there are N singular points, there are at most MN singular points of n^ϵ in Ω with MN independent of ϵ. $\Box$

3.3. **Gradient Estimates.** In this section, we use the three step compactness method to prove L^∞ estimates on gradients of minimizers. In this section, χ_{ki}^β always denote the corrector defined in (3.1). $m_\epsilon^r(x)$ is defined as in (3.24).

Lemma 3.22. *Given $\mu \in (0,1)$, we can find $\theta \in (0,1)$ $\delta_0, \epsilon_0 > 0$ depending only on d, N, μ such that the following statement is true: If n^ϵ is a minimizer of $\int_{B_2(0)} a(\frac{x}{\epsilon})|\nabla n|^2 dx$ with*

$$\int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 \leq \delta_0,$$

then for $\epsilon \leq \epsilon_0$, we have

$$\frac{1}{\theta^{d-2}}\int_{B_\theta(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 dx \leq \theta^{2\mu}\int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 dx. \tag{3.46}$$

and

$$\fint_{B_{\frac{\theta}{2}}(0)} |\nabla n^\epsilon(x) - A^\epsilon_{\theta/2}(x) - \overline{\nabla n^\epsilon}_{B_{\frac{\theta}{2}}}|^2 dx \leq \theta^{2\mu}\fint_{B_{\frac{1}{2}}(0)} |\nabla n^\epsilon(x) - A^\epsilon_{\frac{1}{2}}(x) - \overline{\nabla n^\epsilon}_{B_{\frac{1}{2}}}|^2 dx + \theta^{2\mu}, \tag{3.47}$$

where $A^\epsilon_\theta = (A^{\epsilon\alpha}_{\theta,i}) \in M^{k\times d}$, $A^{\epsilon\alpha}_{\theta,i} = D^\alpha_y \chi^\beta_{ki}(\frac{x}{\epsilon})\overline{\frac{\partial n^\epsilon}{\partial x^\beta}}_{B_\theta}$.

Proof: (3.46) follows from lemma 3.18. We prove (3.47). If (3.47) fails, then for any fixed $\mu, \theta \in (0,1), \delta > 0$ which will be chosen later, there would exist $\epsilon_k \downarrow 0$ and n^{ϵ_k} such that n^{ϵ_k} is a minimizer of $\int_{B_2(0)} a(\frac{x}{\epsilon_k})|\nabla n|^2 dx$ with

$$\int_{B_1(0)} a\left(\frac{x}{\epsilon_k}\right)|\nabla n^{\epsilon_k}|^2 \leq \delta,$$

but

$$\fint_{B_{\frac{\theta}{2}}(0)} |\nabla n^{\epsilon_k} - A^{\epsilon_k}_{\frac{\theta}{2}} - \overline{\nabla n^{\epsilon_k}}_{B_{\frac{\theta}{2}}(0)}|^2 dx > \theta^{2\mu}\fint_{B_{\frac{1}{2}}(0)} |\nabla n^{\epsilon_k} - A^{\epsilon_k}_{\frac{1}{2}} - \overline{\nabla n^{\epsilon_k}}_{B_{\frac{1}{2}}}|^2 dx + \theta^{2\mu}. \tag{3.48}$$

From lemma 3.12 and lemma 3.17, we can find a subsequence (we still denote by n^{ϵ_k}) and a minimizing harmonic map $n \in H^1(B_2(0), N)$ such that

$$n^{\epsilon_k} \rightharpoonup n \text{ weakly in } W^{1,2}(B_2(0))$$

and

$$\int_{B_1(0)} a\left(\frac{x}{\epsilon_k}\right)|\nabla n^{\epsilon_k}|^2 \to \int_{B_1(0)} a_0|\nabla n|^2,$$
$$\int_{B_\theta(0)} a\left(\frac{x}{\epsilon_k}\right)|\nabla n^{\epsilon_k}|^2 \to \int_{B_\theta(0)} a_0|\nabla n|^2,$$
$$\int_{B_{\frac{\theta}{2}}(0)} |\nabla n^{\epsilon_k} - \nabla n - \nabla_y\chi\left(\frac{x}{\epsilon}\right)\nabla n|^2 \to 0,$$
$$\int_{B_{\frac{1}{2}}(0)} |\nabla n^{\epsilon_k} - \nabla n - \nabla_y\chi\left(\frac{x}{\epsilon}\right)\nabla n|^2 \to 0.$$

On the other hand, from the small energy estimates for minimizing harmonic maps, we know there exists $\delta_0 > 0$ depending only on d, N such that if n is a minimizing harmonic map and

$$\int_{B_1(0)} a_0|\nabla n|^2 \leq \delta_0,$$

then $n \in C^\infty(B_{\frac{1}{2}}(0))$. In particular, we have

$$\fint_{B_{\frac{\theta}{2}}(0)} |\nabla n - \overline{\nabla n}_{B_{\frac{\theta}{2}}(0)}|^2 \leq C\theta^2 \fint_{B_{\frac{1}{2}}(0)} |\nabla n - \overline{\nabla n}_{B_{\frac{1}{2}}(0)}|^2.$$

Note

$$\begin{aligned}
&\fint_{B_{\frac{\theta}{2}}(0)} |\nabla n^{\epsilon_k} - \overline{\nabla n^{\epsilon_k}}_{B_{\frac{\theta}{2}}} - \nabla_y \chi\left(\frac{x}{\epsilon}\right) \overline{\nabla n^{\epsilon_k}}_{B_{\frac{\theta}{2}}}|^2 \\
&\leq \fint_{B_{\frac{\theta}{2}}(0)} |\nabla n^{\epsilon_k} - \overline{\nabla n^{\epsilon_k}}_{B_{\frac{\theta}{2}}} - \nabla_y \chi\left(\frac{x}{\epsilon}\right) \nabla n|^2 + C \fint_{B_{\frac{\theta}{2}}(0)} |\nabla n - \overline{\nabla n^{\epsilon_k}}_{B_{\frac{\theta}{2}}}|^2 \\
&\leq \fint_{B_{\frac{\theta}{2}}(0)} |\nabla n - \overline{\nabla n}_{B_{\frac{\theta}{2}}(0)}|^2 + C \fint_{B_{\frac{\theta}{2}}(0)} |\nabla n - \overline{\nabla n}_{B_{\frac{\theta}{2}}(0)}|^2 + O(\epsilon_k) \\
&\leq C\theta^2 \fint_{B_{\frac{1}{2}}(0)} |\nabla n - \overline{\nabla n}_{B_{\frac{1}{2}}}|^2 + C\theta^2 + O(\epsilon_k) \\
&\leq C\theta^2 \fint_{B_{\frac{1}{2}}(0)} |\nabla n^{\epsilon_k} - \overline{\nabla n^{\epsilon_k}}_{B_{\frac{1}{2}}} - \nabla_y \chi\left(\frac{x}{\epsilon}\right) \overline{\nabla n^{\epsilon_k}}_{B_{\frac{1}{2}}}|^2 + C\theta^2 + O(\epsilon_k).
\end{aligned}$$

Therefore if we take θ small enough such that $C\theta^2 \leq \theta^{2\mu}$, pass to the limit in (3.48), a contradiction follows.

Lemma 3.23. *Let $\mu, \theta, \epsilon_0, \delta_0$ be as in lemma 3.22. Suppose n^ϵ is a minimizer of $\int_{B_2(0)} a(\frac{x}{\epsilon})|\nabla n|^2$. If*

$$\int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 \leq \delta_0,$$

then for all k satisfying $\frac{\epsilon}{\theta^k} \leq \epsilon_0$,

$$\frac{1}{\theta^{(d-2)k}} \int_{B_{\theta^k}(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 \leq \theta^{2k\mu} \int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 \tag{3.49}$$

and

$$\begin{aligned}
&\fint_{B_{\frac{\theta^k}{2}}(0)} |\nabla n^\epsilon - \nabla_y \chi\left(\frac{x}{\epsilon}\right) \overline{\nabla n^\epsilon}_{B_{\frac{\theta^k}{2}}} - \overline{\nabla n^\epsilon}_{B_{\frac{\theta^k}{2}}}|^2 dx \\
&\leq \theta^{2k\mu} \fint_{B_{\frac{1}{2}}(0)} |\nabla n^\epsilon - \nabla_y \chi\left(\frac{x}{\epsilon}\right) \overline{\nabla n^\epsilon}_{B_{\frac{1}{2}}} - \overline{\nabla n^\epsilon}_{B_{\frac{1}{2}}}|^2 dx + \theta^{2\mu} \frac{1-\theta^{2k\mu}}{1-\theta^{2\mu}}.
\end{aligned} \tag{3.50}$$

Proof: The proof is by induction on k. $k = 1$ is exactly the conclusion of previous lemma. Now let k satisfying $\epsilon/\theta^k \leq \epsilon_0$ and suppose (3.49) and (3.50) hold. Define

$$w_\epsilon(z) = n^\epsilon(\theta^k z). \tag{3.51}$$

Then w^ϵ is a minimizer of $\int_{B_2(0)} a\left(\frac{\theta^k}{\epsilon}\right)|\nabla w|^2$ satisfying

$$\int_{B_1(0)} a\left(\frac{\theta^k x}{\epsilon}\right)|\nabla w^\epsilon|^2 = \frac{1}{\theta^{(d-2)k}}\int_{B_{\theta^k}(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 \le \theta^{2k\mu}\int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 \le \delta_0.$$

Apply lemma 3.22 to w^ϵ, we obtain

$$\frac{1}{\theta^{d-2}} \fint_{B_\theta(0)} a\left(\frac{\theta^k x}{\epsilon}\right)|\nabla w^\epsilon|^2 \le \theta^{2\mu}\int_{B_1(0)} a\left(\frac{\theta^k x}{\epsilon}\right)|\nabla w^\epsilon|^2 \tag{3.52}$$

and

$$\fint_{B_{\frac{\theta}{2}}(0)} |\nabla w^\epsilon - \nabla_y\chi\left(\frac{\theta^k x}{\epsilon}\right)\overline{\nabla w^\epsilon}_{B_{\frac{\theta}{2}}} - \overline{\nabla w^\epsilon}_{B_{\frac{\theta}{2}}}|^2$$
$$\le \fint_{B_{\frac{1}{2}}(0)} |\nabla w^\epsilon - \overline{\nabla w^\epsilon}_{B_{\frac12}} - \nabla_y\chi\left(\frac{\theta^k x}{\epsilon}\right)\overline{\nabla w^\epsilon}_{B_{\frac12}}|^2 + \theta^{2\mu} \tag{3.53}$$

Rewrite (3.52) and (3.53) utilizing (3.51) and (3.49), we have

$$\frac{1}{\theta^{(d-2)(k+1)}}\int_{B_{\theta^k}(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 \le \theta^{2(k+1)\mu}\int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2$$

and

$$\fint_{B_{\frac{\theta^{k+1}}{2}}(0)} |\nabla n^\epsilon(x) - \nabla_y\chi\left(\frac{x}{\epsilon}\right)\overline{\nabla n^\epsilon}_{B_{\frac{\theta^{k+1}}{2}}} - \overline{\nabla n^\epsilon}_{B_{\frac{\theta^{k+1}}{2}}}|^2$$
$$\le \theta^{2(k+1)\mu}\fint_{B_{\frac12}(0)} |\nabla n^\epsilon - \nabla_y\chi\left(\frac{x}{\epsilon}\right)\overline{\nabla n^\epsilon}_{B_{\frac12}} - \overline{\nabla n^\epsilon}_{B_{\frac12}}|^2 + \theta^{2\mu}\frac{1-\theta^{(2k+1)\mu}}{1-\theta^{2\mu}}$$

The next lemma constitutes a priori interior Lipschitz estimate for minimizers of $\int_\Omega a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2$. For simplicity, we state it for minimizers on $B_1(0)$, the most general case will follow by localization and scaling arguments.

Lemma 3.24. *There exists $\delta_0 > 0$ depending only on d, N satisfies the following: if n^ϵ is a minimizer of $\int_{B_2(0)} a(\frac{x}{\epsilon})|\nabla n|^2$ satisfying*

$$\int_{B_1(0)} a\left(\frac{x}{\epsilon}\right)|\nabla n^\epsilon|^2 dx \le \delta_0,$$

then there exists a constant C depending only on d, N, μ such that

$$|\nabla n^\epsilon|_{L^\infty(B_{\frac12}(0))} \le C\left(\int_{B_1(0)} |\nabla n^\epsilon|^2\right)^{\frac12}$$

Proof: We denote by C a generic constant depending on d, N possibly changing from one estimate to another. Let k be such that

$$\epsilon/\theta^k \le \epsilon_0 < \epsilon/\theta^{k+1}.$$

Substituting this k into lemma 3.23 we see that

$$\fint_{B_{\frac{\epsilon}{\epsilon_0}}(0)} |\nabla n^\epsilon - \nabla_y\chi\left(\frac{x}{\epsilon}\right)\overline{\nabla n^\epsilon}_{B_{\frac{\theta^k}{2}}} - \overline{\nabla n^\epsilon}_{B_{\frac{\theta^k}{2}}}|^2$$
$$\le C\left(\frac{\epsilon}{\epsilon_0}\right)^{2k\mu}\fint_{B_{\frac12}} |\nabla n^\epsilon - \nabla_y\chi\left(\frac{x}{\epsilon}\right)\overline{\nabla n^\epsilon}_{B_{\frac12}} - \overline{\nabla n^\epsilon}_{B_{\frac12}}|^2 + C.$$

From which it follows

$$\fint_{B_{\frac{\epsilon}{\epsilon_0}}(0)} |\nabla n^\epsilon - \overline{\nabla n^\epsilon}_{B_{\frac{\epsilon}{\epsilon_0}}}|^2 \le C\left(\frac{\epsilon}{\epsilon_0}\right)^{2\mu} \fint_{B_{\frac{1}{2}}(0)} |\nabla n^\epsilon - \overline{\nabla n^\epsilon}_{B_{\frac{1}{2}}}|^2 + C(\fint_{B_{\frac{1}{2}}(0)} \nabla n^\epsilon)2$$
$$\le C_1\epsilon^{2\mu} + C_2. \tag{3.54}$$

Let $w^\epsilon(x) = n^\epsilon(\epsilon x)$, then w^ϵ is a minimizer of $\int_{B_2(0)} a_{\alpha\beta}D^\alpha n \cdot D^\beta n$. Rescaling (3.54) we have

$$\fint_{B_{\frac{1}{\epsilon_0}}(0)} |\nabla w^\epsilon - \overline{\nabla w^\epsilon}|^2 \le C_1\epsilon^{2+2\mu} + C_2\epsilon^2.$$

From Lipschitz estimates for minimizers of $\int_{B_1(0)} a(x)|\nabla n|^2$, we know

$$\begin{aligned} |\ \nabla w^\epsilon\ |_{L^\infty(B_{\frac{1}{2\epsilon_0}}(0))} &\le C(\fint_{B_{\frac{1}{\epsilon_0}}(0)} |\nabla w^\epsilon - \overline{\nabla w^\epsilon}|^2 + (\fint_{B_{\frac{1}{\epsilon_0}}(0)} \nabla w^\epsilon)^2)^{\frac{1}{2}} \\ &\le C\epsilon^{2+2\mu} + C\epsilon^2 \end{aligned} \tag{3.55}$$

Rewrite (3.3) in terms of n^ϵ, we have

$$|\nabla n^\epsilon|_{L^\infty(B_{\frac{\epsilon}{\epsilon_0}}(0))} \le C\left(\int_{B_1(0)} |\nabla n^\epsilon|^2\right)^{\frac{1}{2}}.$$

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Final version for publication September 2002.