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A LIOUVILLE-TYPE THEOREM FOR HIGHER ORDER ELLIPTIC SYSTEMS OF HÉ NON-LANE-EMDEN TYPE

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Abstract. We prove there are no positive solutions with slow decay rates to higher order elliptic system

$$\begin{cases} (-\Delta)^m u = |x|^a v^p \\ (-\Delta)^m v = |x|^b u^q \end{cases} \text{ in } \mathbb{R}^N$$

if $p \geq 1$, $q \geq 1$, $(p,q) \neq (1,1)$ satisfies $\frac{1+\frac{a}{N}}{p+1} + \frac{1+\frac{b}{N}}{q+1} > 1 - \frac{2m}{N}$ and

$$\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) > N-2m-1.$$

Moreover, if $N = 2m+1$ or $N = 2m+2$, this system admits no positive solutions with slow decay rates if $p \geq 1$, $q \geq 1$, $(p,q) \neq (1,1)$ satisfies $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N}$.

1. **Introduction.** In this paper, we consider positive solutions $(u > 0,\ v > 0)$ of the following higher order Hénon-Lane-Emden type elliptic system

$$\begin{cases} (-\Delta)^m u = |x|^a v^p \\ (-\Delta)^m v = |x|^b u^q \end{cases} \text{ in } \mathbb{R}^N, \tag{1}$$

where $p > 0$, $q > 0$, $a \geq 0$, $b \geq 0$ and $N \geq 3$. We are mainly concerned with the question of nonexistence of such positive solutions.

The Hénon-Lane-Emden conjecture for polyharmonic system (1) states the following:

Conjecture 1. *Let (u,v) be a pair of nonnegative solution of (1). If*

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2m,$$

then $u = v = 0$.

For $1 \leq N \leq 2m$, the conjecture follows directly from a growth estimate of integral of $|x|^a v^p$ and $|x|^b u^q$ on ball of radius R (Lemma 1 of [4]). We shall focus on cases $N \geq 2m+1$ in this paper. For the rest of the introduction, we shall review some known results for case $a = b = 0$ and for case when at least one of a or b is positive.

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1.1. **Case $a = b = 0$.** When $a = b = 0$. (1) reduces to the well studied Lane-Emden system

$$\begin{cases} (-\Delta)^m u = v^p \\ (-\Delta)^m v = u^q \end{cases} \quad \text{in } \mathbb{R}^N. \tag{2}$$

The conjecture then states that the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2m}{N}$ is the dividing curve for existence and nonexistence of positive solutions of (2) .

For $m = 1$,the conjecture was completely solved in the case of radial solutions [9, 14, 16]. Mitidieri [9] showed that there is no positive radial solutions to (2) below the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ if $p > 1$, $q > 1$; the condition $p > 1$, $q > 1$ was later relaxed to $p > 0$, $q > 0$ by Serrin and Zou [14, 16]. Furthermore, it is proved by Serrin and Zou [16] that there are infinitely many positive radial solutions above the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. Therefore $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ serves as the dividing curve for existence and nonexistence of positive radial solutions of (2) .

The question for the general positive solution to (2), to the best of our knowledge, has not been completely solved yet for $n > 5$. Partial answers have been given over the years. Souto [18] proved nonexistence of positive C^2 solutions below curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N-1}$ when p, $q > 0$. Felmer and de Figureiredo [6] showed that when $0 < p, q \leq \frac{N+2}{N-2}$ and $(p, q) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right)$, (2) has no positive C^2 solutions. Further evidence supporting the conjecture can be found in [10], where it is shown that there exists no positive supersolutions to (2) below the curve

$$\left\{p > 0, q > 0 : \ \frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N-2} \max\left(\frac{1}{p+1}, \frac{1}{q+1}\right)\right\}. \tag{3}$$

We refer to (3) as S curve and the hyperbola in the conjecture $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ will be referred as Sobolev's hyperbola throughout the paper. For $0 < p, q$, if $pq \leq 1$ or $pq > 1$ and $\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \geq N - 2$, nonexistence of positive solutions was proved by Serrin and Zou in [15]. Direct calculation shows this is the same range of (p, q) as region below and on S curve. Furthermore, Serrin and Zou [15] showed (2) admits no positive solutions satisfying algebraic growth at infinity below the Sobolev hyperbola when $N = 3$. For the special case $\min(p, q) = 1$, the conjecture was proved by C.-S. Lin [7]. Busca and Manásevich [2] proved that if $p, q > 0$, $pq > 1$,

$$\frac{N-2}{2} \leq \min\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \leq \max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) < N - 2,$$

and

$$\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \neq \left(\frac{N-2}{2}, \frac{N-2}{2}\right),$$

there exists no positive classical solutions to (2) . Most recently, the conjecture was fully solved in the case $N = 3$ by Poláčik, Quittner and Souplet [13] and by Souplet [17] when $N = 4$. Souplet also proved the conjecture when $N \geq 5$ under the additional assumption that $\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) > N - 3$.

Comparing to the Lane-Emden system for $m = 1$, less is known about the higher order system (2) when $m > 1$. In the single equation case, Mitidieri [9] proved that for $1 < q < \frac{N+4m}{N-4m}$, $N > 4m$, the problem

$$\begin{cases} \Delta^{2m} u = u^q \\ (-\Delta)^s u \geq 0, \ s = 1, 2 \cdots, 2m-1 \end{cases}$$

in $\mathbb{R}^N$ has no positive radial solution of class $C^{4m}\left(\mathbb{R}^N\right)$. For the system case, it is proved in [8, 20] that if $N \geq 3$, $N > 2m$, if $p \geq 1$, $q \geq 1, (p,q) \neq (1,1)$ satisfying

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N}, \tag{4}$$

then system (2) has no positive radial solutions. For general solutions, the results in [8, 20] show that if $p, q \geq 1$, $(p,q) \neq (1,1)$ satisfies

$$\max\left(\frac{2m\,(p+1)}{pq-1}, \frac{2m\,(q+1)}{pq-1}\right) \geq N - 2m,$$

then system (2) admits no positive solutions. It is also proved in [8] that system (2) does not admit any positive solutions if

$$1 < p, q < \frac{N+2m}{N-2m}.$$

Under the additional assumptions $(-\Delta)^i u > 0, (-\Delta)^i v > 0$ for $i = 1, 2 \cdots, m-1$, Yan [20] proved system (2) admits no positive solutions if $pq \leq 1$. Most recently, Arthur, Yan and Zhao [1] proved there are no positive solutions for (2) if $p \geq 1$, $q \geq 1$, $pq > 1$ satisfies (4) when $N = 2m+1$, or $N = 2m+2$. They also proved the conjecture for same p, q under additional assumption $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N - 2m - 1$, therefore generalized Souplet's result to $m \geq 1$.

1.1.1. *The case $a \neq 0$ and or $b \neq 0$.* Liouville type theorem for (1) was first approached by Phan and Souplet [12]. Combining a measure and feedback argument with Pohozaev identity, they proved nonexistence of positive bounded solution to scalar Hénon equation

$$-\Delta u = |x|^a u^p \text{ in } \mathbb{R}^3$$

when $1 < p < 5 + 2a$ and $a > -2$, confirming the conjecture in the case $N = 3, m = 1, a = b > -2$ and $p = q > 1$. Another result confirming the conjecture in scalar case was proved by Cowan [3] where he showed nonexistence of positive bounded solutions for $m = 2$, $N = 5$ provided $1 < p < 9 + 2a$. Phan and Souplet's result was generalized to polyharmonic system (1) when $m = 1$ by Fazly and Ghoussoub ([5]) in dimension 3 and for $m \geq 1$ by Fazly [4] in dimension $N = 2m+1$. Fazly also shows that (1) does not admit any positive solution (u, v) if $\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) > N - 2m$. In fact, it is pointed out in [11] that (1) does not admit any positive solution (u, v) if $\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) \geq N - 2m$ by a similar argument as in [15]. Moreover, the following theorems are proved by Phan when $m = 1$.

Theorem 1.1 (Theorem 1.1 [11])**.** *Let $a, b > -2$ and $N \geq 3$. Assume $pq > 1$, $p \geq q$. Assume*

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2. \tag{5}$$

Assume in addition that

$$0 \leq a - b \leq \frac{N-2}{p-q},$$

$$\max\left(\frac{2\,(p+1)}{pq-1}, \frac{2\,(q+1)}{pq-1}\right) > N - 3.$$

Then (1) with $m = 1$ has no positive solution.

Theorem 1.2 (Theorem 1.2 [11]). *Let $a, b > -2$ and $N \geq 3$. Assume $pq > 1$, $p \geq q$. Assume (5) and*

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2. \tag{6}$$

Assume in addition that

$$\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) > N - 3.$$

Then (1) with $m = 1$ has no positive solution.

For case $a < 0$, $b < 0$, Liouville type theorems for both integer and fractional Laplacian have been obtained in [19].

In this paper, we prove the following Liouville type theorems for (1).

Theorem 1.3. *$N \geq 3$, $N > 2m$, $a \geq 0$, $b \geq 0$, assume $p \geq 1$, $q \geq 1$, $(p, q) \neq (1, 1)$. We have the following Liouville type result: If*

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2m \tag{7}$$

and

$$\max\left(\frac{2m(p+1) + a + bp}{pq-1}, \frac{2m(q+1) + aq + b}{pq-1}\right) > N - 2m - 1,$$

the problem (1) has no positive solutions of class $C^{2m}\left(\mathbb{R}^N\right)$ which satisfies slow decay assumptions

$$u(x) \leq C \min\left(|x|^{-\alpha}, 1\right), \; v(x) \leq C \min\left(|x|^{-\beta}, 1\right) \tag{8}$$

Moreover, when $N = 2m + 1$ or $2m + 2$, if $p \geq 1, q \geq 1$, $(p, q) \neq (1, 1)$ satisfies (7), (1) admits no positive solutions satisfying (8).

Under stronger assumptions on p, q, we can remove the decay assumptions on (u, v).

Theorem 1.4. *$N \geq 3$, $N > 2m$, if $p \geq 1$, $q \geq 1$, $(p, q) \neq (1, 1)$ satisfies*

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2m \tag{9}$$

and

$$\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N - 2m - 1,$$

the problem (1) has no positive solutions of class $C^{2m}\left(\mathbb{R}^N\right)$. Moreover, when $N = 2m + 1$ or $2m + 2$, if $p \geq 1, q \geq 1$, $(p, q) \neq (1, 1)$ satisfies (9), then (1) admits no positive solutions.

The paper is organized as follows. Section 2 presents some technical Lemmas as preparation, section 3 is devoted to the proof of Theorem 1.3 and Theorem 1.4. Our proof of Theorem 1.3 relies on a Rellich-Pohozaev identity combined with an adapted idea of measure and feedback argument in Souplet's paper [17]. Proof of Theorem 1.4 is based on an adapted idea of a doubling property Lemma from [13] and Liouville theorems for (2).

2. **Preparations.** When $pq > 1$,we introduce the following notations

$$\alpha = \frac{2m\left(p+1\right)+a+bp}{pq-1}, \quad \beta = \frac{2m\left(q+1\right)+aq+b}{pq-1}$$

and assume $\alpha \geq \beta$ throughout the rest of the paper. The assumption

$$\frac{1+\frac{a}{N}}{p+1} + \frac{1+\frac{b}{N}}{q+1} > \frac{N-2m}{N}$$

can be rewritten as

$$\alpha + \beta > N - 2m.$$

For $w \in C\left(\mathbb{R}^N\right)$, we denote the spherical average of w by

$$\overline{w}\left(r\right) = \frac{1}{\omega_N}\int_{S^{N-1}} w\left(r,\theta\right) ds, \quad r > 0,$$

where ω_N is the area of the unit sphere S^{N-1}.

We have following growth estimates.

Lemma 2.1. *If $pq = 1$, there is no positive solution of (1). If (u, v) is a positive solution of (1) and $p, q \geq 1$, and $pq > 1$, there exists a positive constant $M = M\left(p,q,n\right)$ such that*

$$\overline{u}\left(r\right) \leq Mr^{-\alpha}, \qquad \overline{v}\left(r\right) \leq Mr^{-\beta} \quad \text{for} \quad r > 0. \tag{10}$$

and for $k = 1, \cdots, m-1$, $u_k = (-\Delta)^k u, v_k = (-\Delta)^k v$, we have

$$(-\Delta)^i u > 0, \quad (-\Delta)^i v > 0, \ i = 1, 2, \cdots, m-1.$$

$$\overline{u_k}\left(r\right) \leq Mr^{-\alpha-2k}, \qquad \overline{v_k}\left(r\right) \leq Mr^{-\beta-2k} \quad \text{for} \quad r > 0. \tag{11}$$

Proof. Lemma follows from the same argument as in proof of Lemma 3.3 in [20]. □

The following growth estimates was proved in [4].

Lemma 2.2 (Lemma 1 in [4]). *Suppose that $p, q \geq 1$ and (u, v) is a positive solution of (1). Then*

$$\int_{B_R} |x|^b u^q \leq cR^{N-2m-\beta}, \quad \int_{B_R} |x|^a v^p \leq cR^{N-2m-\alpha}, \tag{12}$$

where $c = c\left(p,q,n\right)$.

As a direct corollary of Lemma 2.2, we have the following nonexistence result for (1). This was pointed out in [11] We write down the details for readers' convenience.

Corollary 1. *If $p, q \geq 1$ and $\max\left(\alpha,\beta\right) \geq N - 2m$, (1) does not admit any positive solution.*

Proof. We only need to prove case $\max\left(\alpha,\beta\right) = N - 2m$. Without loss of generality, we can assume $\alpha \geq \beta$. Recall that for $w > 0$, $\Delta w \leq 0$, we have

$$w\left(x\right) \geq c\left|x\right|^{2-N} \ \text{ for } \ |x| \geq 1.$$

Since

$$-\Delta u_{k-1} = u_k,$$

it follows from Lemma 2.7 of [15] that

$$\overline{u_{k-1}} \geq cr^2\overline{u_k}, \quad k = 1, \cdots, m-1.$$

Iteration then gives

$$\overline{u}\left(r\right) \geq r^{2m-N} \ \text{ for } r \geq 1.$$

Applying Lemma 2.7 of [15] to $\overline{v_k}$ for $k = 0, 1, \cdots m-1$ yields

$$\overline{v}(r) \geq C r^{2m+b} \overline{u^q} \geq C r^{2m+b} \overline{u}^q \geq C r^{2m+b-(N-2m)q}.$$

Therefore by (12)

$$\begin{aligned} C &\geq \int_{B_R} |x|^a v^p \geq \int_0^R r^{N-1+a} \overline{v}^p \\ &\geq \int_1^R r^{N-1+a+2mp+bp-(N-2m)pq} = \int_1^R r^{-1} dr = \ln R \end{aligned} \tag{13}$$

The first equality in (13) follows from assumption on α and identity

$$N - 1 + a + 2mp + bp - (N-2m)pq = -1 + (pq-1)(\alpha - N + 2m) = -1.$$

Letting R goes to infinity in (13), contradiction. $\square$

We state the following interpolation inequalities and elliptic estimates.

Lemma 2.3 (L^p estimates on B_R)**.** *Given* $1 < k < \infty$, $R > 0$, $z \in W^{2m,k}(B_{2R})$, *then*

$$\int_{B_R \setminus B_{\frac{R}{2}}} \left|D^{2m} z\right|^k \leq C \left(\int_{B_{2R} \setminus B_{\frac{R}{4}}} |\Delta^m z|^k + R^{-2mk} \int_{B_{2R} \setminus B_{\frac{R}{4}}} |z|^k \right).$$

Proof. Lemma follows from standard elliptic L^p estimates for elliptic equations and interpolation inequalities. $\square$

Lemma 2.4. *For any* $R > 0$, $l = 1, 2, \cdots m-1$,

$$\int_{B_R} |\nabla_x u_l| \leq CR \int_{B_{2R}} |u_{l+1}| + CR^{-1} \int_{B_{2R}} |u_l|.$$

Lemma 2.5. *(Sobolev inequality on S^{N-1})* $N \geq 2$, $j \geq 1$, $1 < \mu < \lambda \leq \infty$. $\mu \neq \frac{N-1}{j}$

$$\|w\|_\lambda \leq C \left(\left\| D_\theta^j w \right\|_\mu + \|w\|_1 \right),$$

here

$$\begin{aligned} \frac{1}{\mu} - \frac{1}{\lambda} &= \frac{j}{N-1} \quad \text{if } \mu < \frac{N-1}{j}, \\ \lambda &= \infty \quad \text{if } \mu > \frac{N-1}{j}. \end{aligned}$$

We can prove the following growth estimates for u, v and their derivatives.

Lemma 2.6. *Let*

$$\begin{aligned} k &= \frac{2m(p+1)(q+1) + a(q+1) + b(p+1)}{2mp(q+1) + a + bp}, \\ d &= \frac{2m(p+1)(q+1) + a(q+1) + b(p+1)}{2mq(p+1) + aq + b}. \end{aligned}$$

If bounded solution pair (u, v) *of* (1)*satisfies the following decay assumptions*

$$u(x) \leq C|x|^{-\alpha}, \; v(x) \leq C|x|^{-\beta} \quad \text{for } |x| \geq 1, \tag{14}$$

then the following estimates hold for $l = 1, 2 \cdots, m-1$,

$$\int_0^R \|u_l(r)\|_1 r^{N-1} dr \leq C r^{N-\alpha-2l}, \tag{15}$$

$$\int_0^R \|v_l(r)\|_1 r^{N-1} dr \leq C r^{N-\beta-2l}, \tag{16}$$

$$\int_0^R \|D_x u_l\|_1 r^{N-1} dr \leq C r^{N-\alpha-2l-1}, \tag{17}$$

$$\int_0^R \|D_x v_l\|_1 r^{N-1} dr \leq C r^{N-\beta-2l-1}, \tag{18}$$

$$\int_{\frac{R}{2}}^R \left\|D_x^{2m} u\right\|_k^k r^{N-1} dr \leq C F(2R), \tag{19}$$

$$\int_{\frac{R}{2}}^R \left\|D_x^{2m} v\right\|_d^d r^{N-1} dr \leq C F(2R), \tag{20}$$

$$\int_0^R \left\|D_x^{2m} u\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \leq C R^{N-2m-\alpha+a\epsilon}, \tag{21}$$

$$\int_0^R \left\|D_x^{2m} v\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \leq C R^{N-2m-\beta+b\epsilon}. \tag{22}$$

Here

$$F(R) = \int_{B_R} \left[|x|^a v^{p+1} + |x|^b u^{q+1}\right] dx.$$

Proof. $(15), (16)$ are restatements of Lemma 2.1. (17) and (18) follows directly from Lemma 2.1 and Lemma 2.4. To prove (19), Lemma 2.3 implies

$$\begin{aligned}
\int_{R/2}^R \left\|D_x^{2m} u\right\|_k^k r^{N-1} dr =& \int_{B_R \setminus B_{R/2}} \left|D^{2m} u\right|^k \\
\leq& C\left(\int_{B_{2R} \setminus B_{R/4}} |\Delta^m u|^k + R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k\right) \\
=& C\left(\int_{B_{2R} \setminus B_{R/4}} |x|^{ak} v^{pk} + R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k\right) \\
\leq& C\left(\int_{B_{2R}} |x|^a v^{p+1} + R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k\right).
\end{aligned}$$

Here we used growth assumption (14) and identity

$$\frac{a(k-1)}{pk-(p+1)} = \beta.$$

Since $pq > 1$, it follows $\frac{p+1}{p} < q+1$ therefore $\frac{p+1}{p} < k < q+1$. By Hölder's inequality and the fact that $F(R) \geq F(1) > 0$, $R \geq 1$, we obtain

$$\begin{aligned}
&R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k \\
\leq& CR^{-2mk} \left(\int_{B_{2R}} |x|^b u^{q+1} \right)^{\frac{k}{q+1}} \left(\int_{B_{2R} \setminus B_{R/4}} |x|^{-\frac{bk}{q+1-k}} \right)^{1-\frac{k}{q+1}} \\
\leq& CR^{-2mk} F(2R)^{\frac{k}{q+1}} R^{\frac{N(q+1-k)-bk)}{q+1}} \\
\leq& CR^{\chi k} F(2R) (F(1))^{\frac{k}{q+1}-1} \\
\leq& CR^{\chi k} F(2R),
\end{aligned}$$

where

$$\chi = -2m - \frac{N+b}{q+1} + \frac{N}{k}.$$

We can write

$$\begin{aligned}
&\chi [2m(p+1)(q+1) + a(q+1) + b(p+1)] \\
=& 2m(pq-1)[N - 2m - \alpha - \beta] + b(p+1) \left[N - 2m - \frac{N+a}{p+1} - \frac{N+b}{q+1} \right]
\end{aligned}$$

Since

$$\frac{1+\frac{a}{N}}{p+1} + \frac{1+\frac{b}{N}}{q+1} > 1 - \frac{2m}{N},$$

we have $\chi < 0$, and (19) follows. (20) is proved similarly by using (14) and

$$\frac{b(d-1)}{qd-(q+1)} = \alpha$$

Lastly we prove (21).

$$\begin{aligned}
\int_0^R \left\| D_x^{2m} u \right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \leq& C \left(\int_{B_{2R}} |\Delta^m u|^{1+\varepsilon} + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u^{1+\varepsilon} \right) \\
=& C \left(\int_{B_{2R}} |x|^{a(1+\epsilon)} v^{p(1+\varepsilon)} + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u^{1+\varepsilon} \right) \\
\leq& C \left(R^{a\epsilon} \int_{B_{2R}} |x|^a v^p + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u \right) \\
\leq& C \left(R^{N-2m-\alpha+a\epsilon} + R^{-2m(1+\varepsilon)} \cdot R^{N-\alpha} \right) \\
\leq& CR^{N-2m-\alpha+a\epsilon}.
\end{aligned}$$

□

In the rest of the section, we prove a Rellich-Pohozaev identity.
We recall the following function defined in [9]

$$R_n(u, v) = \int_\Omega [\Delta^n u(x, \nabla v) + \Delta^n v(x, \nabla u)] \, dx,$$

where $\Omega \subset \mathbb{R}^N$, $u, v \in C^{2n}\left(\overline{\Omega}\right), n \geq 1$. If $n = 1$, we have

$$R_1(u,v) = \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial n}(x, \nabla v) + \frac{\partial v}{\partial n}(x, \nabla u) - (\nabla u, \nabla v)(x, n) \right\} ds + (N-2) \int_{\Omega} (\nabla u, \nabla v)\, dx.$$

If $n = 2$,

$$R_2(u,v) = R_1(\Delta u, v) + R_1(u, \Delta v) - B(u,v), \tag{23}$$

where

$$B(u,v) = \int_{\partial\Omega} \Delta u \Delta v\,(x,n)\, ds - N \int_{\Omega} \Delta u \Delta v dx. \tag{24}$$

We quote the following Lemma from [9]

Lemma 2.7 (Lemma 2.2 in [9]). *If $u, v \in C^{2n}\left(\overline{\Omega}\right)$, then for $1 \leq s \leq n-2$*

$$R_n(u,v) = \sum_{k=0}^{s} R_{n-s}\left(\Delta^k u, \Delta^{s-k} v\right) - \sum_{k=0}^{s-1} R_{n-(s+1)}\left(\Delta^{k+1} u, \Delta^{s-k} v\right). \tag{25}$$

Remark 1. An immediate consequence of Lemma 2.7 is the following implicit form of Rellich's identity. If $u, v \in C^{2n}\left(\overline{\Omega}\right)$, then

$$R_n(u,v) = \sum_{k=0}^{n-1} R_1\left(\Delta^k u, \Delta^{n-1-k} v\right) - \sum_{k=0}^{n-2} B\left(\Delta^k u, \Delta^{n-2-k} v\right). \tag{26}$$

Proof. Choose $s = n-2$ in (25), taking into account of (23) and (24), (26) follows. ∎

Write

$$u^{q+1}(r) = \int_{S^{N-1}} u^{q+1}(r,\theta)\, d\theta, \quad v^{p+1}(r) = \int_{S^{N-1}} v^{p+1}(r,\theta)\, d\theta,$$

we have the following Rellich-Pohozav identity.

Lemma 2.8. *For any $a_1 + a_2 = N - 2m$, $r > 0$*

$$\begin{aligned}
&\left(\frac{N+a}{p+1} - a_1\right) \int_{B_r} |x|^a v^{p+1} dx + \left(\frac{N+b}{q+1} - a_2\right) \int_{B_r} |x|^b u^{q+1} dx \\
=& \frac{1}{p+1} v^{p+1}(r)\, r^{N+a} + \frac{1}{q+1} u^{q+1}(r)\, r^{N+b} \\
&- (-1)^m \left\{ \sum_{k=0}^{m-1} 2r^N \int_{S^{N-1}} \frac{\partial \Delta^k u}{\partial n} \cdot \frac{\partial \Delta^{m-1-k} v}{\partial n} ds \right. \\
&- \sum_{k=0}^{m-1} r^N \int_{S^{N-1}} \left(\nabla \Delta^k u, \nabla \Delta^{m-1-k} v\right) ds - \sum_{k=0}^{m-2} r^N \int_{S^{N-1}} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right) ds \\
&+ \sum_{k=0}^{m-1} (2m - 2k - 2 + a_1)\, r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v ds \\
&\left. + \sum_{l=0}^{m-1} (a_2 + 2k)\, r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^{m-1-k} v}{\partial n} \Delta^k u\ ds \right\}.
\end{aligned}$$

Proof. A similar Rellich-Pohozaev identity can be found in [4]. For purpose of later estimates, we prefer to write our Rellich-Pohozaev identity with a slightly different boundary terms on the right hand side. By (1)

$$
\begin{aligned}
(-1)^m R_m(u,v) &= \int_{B_r} (-\Delta)^m u(x)(x,\nabla v) + (-\Delta)^m v(x,\nabla u)\,dx \\
&= \int_{B_r} v^p(x)(x,\nabla v) + u^q(x)(x,\nabla u)\,dx \\
&= \int_{\partial B_r} \frac{v^{p+1}}{p+1}|x|^a(x,n) + \frac{u^{q+1}}{q+1}|x|^b(x,n)\,ds \\
&\quad - \frac{N+a}{p+1}\int_{B_r}|x|^a v^{p+1}dx - \frac{N+b}{q+1}\int_{B_r}|x|^b u^{q+1}dx \\
&= \frac{1}{p+1}v^{p+1}(r)\,r^{N+a} + \frac{1}{q+1}u^{q+1}(r)\,r^{N+b} \\
&\quad - \frac{N+a}{p+1}\int_{B_r}|x|^a v^{p+1}dx - \frac{N+b}{q+1}\int_{B_r}|x|^b u^{q+1}dx.
\end{aligned}
$$

To finish the proof, we follow the same argument as in proof of Lemma 2.8 in [1] to estimate $R_m(u,v)$ using (26) and integration by parts. $\square$

3. **Proof of the theorems.**

3.1. **Solutions with no decay assumptions.** In this subsection, we prove if the system (2) does not admit bounded positive solution, then (1) with same p,q does not admit classical positive solution. More precisely, we prove the following Theorem. From this theorem, Theorem 1.4 follows.

Theorem 3.1. *Let $N \geq 3$, $p > 1$, $q > 1$ be fixed, and assume (2) does not admit any bounded nontrivial (nonnegative) solution in $\mathbb{R}^N$, then (1) with same p,q does not admit any notrivial (nonnegative) solution in $\mathbb{R}^N$, bounded or not. In particular, the conclusion holds if $N = 2m+1$, or $2m+2$ and $\frac{2m(p+1)}{pq-1} + \frac{2m(q+1)}{pq-1} > N-2m-1$. The conclusion also holds when $N > 2m+2$ and p,q satisfies $\frac{2m(p+1)}{pq-1} + \frac{2m(q+1)}{pq-1} > N-2m-1$ and $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N-2m-1$.*

Proof of Theorems 3.1 use an adapted idea of [13] (see also [12] for this adapted idea for single equation case), which relies on the following Doubling property Lemma and remark.

Lemma 3.2 (Lemma 5.1 [13])**.** *Let (X,d) be a complete metric space, and let $\emptyset \neq D \subset \Sigma \subset X$ with Σ closed. Set $\Gamma = \Sigma \backslash D$. Finally, let $M : D \to (0,\infty)$ be bounded on compact subsets of D, and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)\,dist(y,\Gamma) > 2k,$$

then there exists $x \in D$ such that

$$M(x)\,dist(x,\Gamma) > 2k, \quad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \text{for all } z \in D \cap \overline{B_X}(x, kM^{-1}(x)).$$

Remark 2. (Remark 5.2 [13]).

(a): If $\Gamma=\emptyset$, then $\operatorname{dist}(x,\Gamma)=\infty$.

(b): Take $X=\mathbb{R}^n$, take Ω an open subset of $\mathbb{R}^n$, put $D=\Omega$, $\Sigma=\overline{D}$; hence $\Gamma=\partial\Omega$. Then we have $\overline{B}(x,kM^{-1}(x))\subset D$. Indeed, since D is open, implies $\operatorname{dist}(x,D^c)=\operatorname{dist}(x,\Gamma)>2kM^{-1}(x)$.

We first prove the following Lemma.

Lemma 3.3. *Let $\delta\in(0,1]$. Let $c_i\in C^{\delta}\left(\overline{B_1}\right)$ satisfy*

$$\|c_i\|_{C^{\delta}}\left(\overline{B_1}\right)\le C_1 \quad \text{and } c_i(x)\ge C_2,\quad x\in\overline{B_1}, i=1,2$$

for some positive constants C_1, C_2. Assume (2) does not admit any bounded positive solutions. There exists a constant C, depending only on δ, C_1, C_2, p, q, N, such that any nonnegative solutions (u,v) of

$$\begin{cases} (-\Delta)^{2m}u=c_1(x)v^p \\ (-\Delta)^{2m}v=c_2(x)u^q \end{cases} \quad x\in\overline{B_1} \tag{27}$$

with same p,q satisfies

$$u(x)\le C\left(1+dist^{-\gamma}(x,\partial B_1)\right),\ x\in B_1$$

and

$$v(x)\le C\left(1+dist^{-\sigma}(x,\partial B_1)\right),\ x\in B_1.$$

Here $\gamma=\frac{2m(p+1)}{pq-1}$, $\sigma=\frac{2m(q+1)}{pq-1}$.

Proof. Assume the Lemma fails. Then there exist sequences (u_k,v_k), $y_k\in B_1$ such that (u_k,v_k) solves (27) on B_1 and

$$M_k:=u_k^{\frac{1}{\gamma}}+v_k^{\frac{1}{\sigma}},\quad k=1,2,\cdots$$

satisfies

$$M_k(y_k)>2k\left(1+\operatorname{dist}^{-1}(y_k,\partial B_1)\right).$$

By Lemma 3.2 and Remark 2, it follows that there exists $x_k\in B_1$ such that

$$M_k(x_k)\ge M_k(y_k)>2k$$

and

$$M_k(z)\le 2M_k(x_k),\quad \text{for } |z-x_k|\le kM_k^{-1}(x_k).$$

Define rescaling of (u_k,v_k) as follows

$$\begin{aligned}\lambda_k=&M_k^{-1}(x_k)\\ \widetilde{u}_k(y)=&\lambda_k^{\gamma}u_k(x_k+\lambda_k y),\quad \widetilde{v}_k(y)=\lambda_k^{\sigma}v_k(x_k+\lambda_k y),\quad |y|\le k.\end{aligned}$$

We then have $\lambda_k\to 0$ and $(\widetilde{u}_k,\widetilde{v}_k)$ satisfies

$$\begin{aligned}(-\Delta_y)^m\widetilde{u}_k(y)=&\widetilde{c}_{1k}(y)\widetilde{v}_k^p(y)\\ (-\Delta_y)^m\widetilde{v}_k(y)=&\widetilde{c}_{2k}(y)\widetilde{u}_k^q(y)\end{aligned}$$

for $|y|\le k$. Here

$$\widetilde{c}_{ik}(y)=c_i(x_k+\lambda_k y),\quad i=1,2$$

satisfies $C_2\le\widetilde{c}_{ik}(y)\le C_1$ and for each $R>0$, $k\ge k_0(R)$

$$|\widetilde{c}_{ik}(y)-\widetilde{c}_{ik}(z)|\le C_1|\lambda_k(y-z)|^{\delta}\le C_1|(y-z)|^{\delta}\quad \text{for } |y|,|z|\le R. \tag{28}$$

By Ascoli-Arzelá theorem, there exists $\widetilde{c}_i$ in $c\left(\mathbb{R}^N\right)$ with $\widetilde{c}\ge C_2$ such that $\widetilde{c}_{ik}\to\widetilde{c}_i$ in $C_{loc}\left(\mathbb{R}^N\right)$ subject to a subsequence. Since $\lambda_k\to 0$, (28) implies limit functions $\widetilde{c}_i$ are actually constants. We write the limit constants as l_1, l_2. Moreover, By standard elliptic L_p estimates and Sobolev embeddings, we conclude that subject to a subsequence, $(\widetilde{u}_k, \widetilde{v}_k)$ converges in $C^{2m}_{loc}(\mathbb{R}^N)$ to a (classical) solution $(\widetilde{u}, \widetilde{v})$ of

$$(-\Delta_y)^m \widetilde{u}(y) = l_1 \widetilde{v}^p(y)$$
$$(-\Delta_y)^m \widetilde{v}(y) = l_2 \widetilde{u}^q(y) \tag{29}$$

in $\mathbb{R}^N$. Since

$$\widetilde{u}_k^{\frac{1}{\gamma}}(0) + \widetilde{v}_k^{\frac{1}{\sigma}}(0) = 1$$

and

$$\widetilde{u}_k^{\frac{1}{\gamma}}(y) + \widetilde{v}_k^{\frac{1}{\sigma}}(y) \leq 2, \text{ when } \quad |y| \leq k.$$

We have $\widetilde{u}^{\frac{1}{\gamma}}(0) + \widetilde{v}^{\frac{1}{\sigma}}(0) = 1$ and $\widetilde{u}^{\frac{1}{\gamma}}(y) + \widetilde{v}^{\frac{1}{\sigma}}(y) \leq 2$. i.e. $(\widetilde{u}, \widetilde{v})$ is nontrivial and bounded solution of (29), contradicting the assumption for (2). In particular, Liouville theorems for (2) implies the assumption holds when $N = 2m+1$, or $2m+2$ and $\frac{2m(p+1)}{pq-1} + \frac{2m(q+1)}{pq-1} > N - 2m - 1$. The assumption of this Lemma also holds when $N > 2m+2$ and p, q satisfies $\frac{2m(p+1)}{pq-1} + \frac{2m(q+1)}{pq-1} > N - 2m - 1$ and $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N - 2m - 1$. □

Lemma 3.4. *Assume (2) does not admit any bounded nontrivial nonegative solution in $\mathbb{R}^N$. There exists a constant $C = C(N, p, q, a, b) > 0$ (independent of Ω and (u, v)) such that the following holds.*

i): Any nonnegative solution of (1) in $\Omega = \left\{x \in R^N : 0 < |x| < \rho\right\}$ satisfies

$$u(x) \leq C|x|^{-\alpha} \quad \text{and } v(x) \leq C|x|^{-\beta} \ , \ 0 < |x| < \frac{\rho}{2}.$$

ii): Any nonnegative solution of (1) in $\Omega = \left\{x \in R^N : |x| > \rho\right\}$ satisfies

$$u(x) \leq C|x|^{-\alpha} \text{ and } v(x) \leq C|x|^{-\beta}, \quad |x| > 2\rho.$$

Proof. Assume either $\Omega = \left\{x \in R^N : 0 < |x| < \rho\right\}$ and $0 < |x_0| < \frac{\rho}{2}$ or $\Omega = \left\{x \in R^N : 0 < |x| < \rho\right\}$ and $|x| > 2\rho$. Let $R = \frac{|x_0|}{2}$, it then follows

$$\frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2} \text{ for } y \in B_1.$$

So $x_0 + Ry \in \Omega$ in either case. Define

$$U(y) = R^\alpha u(x_0 + Ry), \quad V(y) = R^\beta v(x_0 + Ry).$$

Then for $y \in B_1$, (U, V) is a solution to

$$\begin{cases} (-\Delta)^{2m} U = c(y)^a V^p(y) \\ (-\Delta)^{2m} V = c(y)^b U^q(y) \end{cases}$$

with $c(y) = \left|y + \frac{x_0}{R}\right|$. Recall that $\left|y + \frac{x_0}{R}\right| \in [1, 3]$ for $y \in B_1$ and $\|c(y)\|_{C^1} \leq C$. Apply Lemma 3.3 we yield

$$U(0) + V(0) \leq C.$$

From which it follows

$$u(x_0) < CR^{-\alpha}, \ v(x_0) < CR^{-\beta},$$

the conclusion then follows. □

Proof of Theorem 3.1. Assume (u, v) is a solution of (1) on $\mathbb{R}^N$ (bounded or not). Then for each $x_0 \in \mathbb{R}^N$ and $R > 0$, by applying Lemma 3.4 in $\Omega = B(x_0, R)$, we obtain

$$u(x_0) \leq CR^{-\alpha}, \quad v(x_0) \leq CR^{-\beta}.$$

Letting $R \to \infty$, we obtain

$$u(x_0) = v(x_0) = 0,$$

therefore

$$u \equiv v \equiv 0.$$

□

3.2. **Proof of theorem 1.3.** We shall adapt Souplet's idea [17] of a measure and feedback argument combined with Rellich-Pohazaev identity. Lemma 2.8 implies

$$F(R) \leq CG_1(R) + CG_2(R),$$

where

$$G_1(R) = R^N \sum_{l=0}^{m} \int_{S^{N-1}} |u_l| \, |v_{m-l}| \, ds$$

and

$$G_2(R) = R^N \int_{S^{N-1}} \sum_{l=0}^{m-1} \left(\left| u'_{m-l-1} \right| + R^{-1} \left| u_{m-l-1} \right| \right) \left(|v'_l| + R^{-1} |v_l| \right) ds.$$

Following Souplet's idea, we shall prove there exist constants C, $a > 0$, $b < 1$ such that

$$F(R) \leq CR^{-a} F^b(R). \tag{30}$$

It then follows

$$F(R) \to 0 \quad \text{as} \quad R \to \infty,$$

which implies

$$u = v \equiv 0.$$

To prove (30), we follow a similar procedure as [17]. We shall first estimate $G_1(R)$ and $G_2(R)$ in terms of highest derivatives of the solution (u, v) and (u, v) in suitable L_p spaces. Then use a feedback and measure argument to evaluate those bounds in terms of $F(R)$.

Step1. *Estimation of* $G_1(R)$ *in terms of suitable norms of* $D_x^{2m} u(R)$ *and* $D_x^{2m} v(R)$.

Fix $l \in \{0, 1, \cdots m\}$, Hölder's inequality gives

$$\int_{S^{N-1}} |u_l| \, |v_{m-l}| \, ds \leq \|u_l\|_{\alpha_l} \|v_{m-l}\|_{\alpha'_l},$$

where $\frac{1}{\alpha_l} + \frac{1}{\alpha'_l} = 1$ is chosen so that

$$\begin{aligned} \frac{1}{k} - \frac{2m-2l}{N-1} &\leq \frac{1}{\alpha_l} \leq 1 - \frac{2m-2l}{N-1}, \\ \frac{1}{d} - \frac{2l}{N-1} &\leq 1 - \frac{1}{\alpha_l} \leq 1 - \frac{2l}{N-1}. \end{aligned} \tag{31}$$

Here

$$\begin{aligned}\frac{1}{k} =&\frac{2mp\left(q+1\right)+a+bp}{2m\left(p+1\right)\left(q+1\right)+a\left(q+1\right)+b\left(p+1\right)},\\ \frac{1}{d} =&\frac{2mq\left(p+1\right)+aq+b}{2m\left(p+1\right)\left(q+1\right)+a\left(q+1\right)+b\left(p+1\right)}.\end{aligned}$$

Such α_l exists since by assumption,

$$\frac{N+a}{p+1}+\frac{N+b}{q+1}>N-2m>N-1-2m.$$

Let

$$\begin{aligned}\frac{1}{\gamma_l} =&\frac{1}{k}-\frac{2m-2l}{N-1},\frac{1}{\delta_l}=\frac{N-2m+2l-1}{N-1},\\ \frac{1}{\omega_l} =&\frac{1}{d}-\frac{2l}{N-1},\frac{1}{\psi_l}=\frac{N-2l-1}{N-1}.\end{aligned}$$

Case I: $\gamma_l>0$, $\omega_l>0$.

By Hölder's inequality, we have

$$\begin{aligned}\left\|u_l\right\|_{\alpha_l}\leq\left\|u_l\right\|_{\delta_l}^{\nu_{1l}}\left\|u_l\right\|_{\gamma_l}^{1-\nu_{1l}},\\ \left\|v_{m-l}\right\|_{\alpha_l'}\leq\left\|v_{m-l}\right\|_{\psi_l}^{\nu_{2l}}\left\|v_{m-l}\right\|_{\omega_l}^{1-\nu_{2l}},\end{aligned}\tag{32}$$

with

$$\frac{1}{\alpha_l}=\frac{\nu_{1l}}{\delta_l}+\frac{1-\nu_{1l}}{\gamma_l},\quad\frac{1}{\alpha_l'}=\frac{\nu_{2l}}{\psi_l}+\frac{1-\nu_{2l}}{\omega_l}.$$

Applying Lemma 2.5, we deduce

$$\left\|u_l\right\|_{\delta_l}\leq C\left(\left\|D_\theta^{2m-2l}u_l\right\|_{1+\varepsilon}+\left\|u_l\right\|_1\right)\leq C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon}+\left\|u_l\right\|_1\right),\tag{33}$$

$$\left\|u_l\right\|_{\gamma_l}\leq C\left(\left\|D_\theta^{2m-2l}u_l\right\|_k+\left\|u_l\right\|_1\right)\leq C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_k+\left\|u_l\right\|_1\right),\tag{34}$$

and

$$\left\|v_{m-l}\right\|_{\psi_l}\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_{1+\varepsilon}+\left\|v_{m-l}\right\|_1\right)\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon}+\left\|v_{m-l}\right\|_1\right),\tag{35}$$

$$\left\|v_{m-l}\right\|_{\omega_l}\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_d+\left\|v_{m-l}\right\|_1\right)\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_d+\left\|v_{m-l}\right\|_1\right).\tag{36}$$

Combining (32), (33), (34), (35) and (36), we conclude

$$\begin{aligned}&\int_{S^{N-1}}\left|u_l\right|\left|v_{m-l}\right|ds\leq\left\|u_l\right\|_{\alpha_l}\left\|v_{m-l}\right\|_{\alpha_l'}\\ \leq&CR^{2m}\left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon}+R^{-2m+2l}\left\|u_l\right\|_1\right)^{\nu_{1l}}\\ &\cdot\left(\left\|D_x^{2m-2l}u_l\right\|_k+R^{-2m+2l}\left\|u_l\right\|_1\right)^{1-\nu_{1l}}\cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon}+R^{-2l}\left\|v_{m-l}\right\|_1\right)^{\nu_{2l}}\\ &\cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_d+R^{-2l}\left\|v_{m-l}\right\|_1\right)^{1-\nu_{2l}}.\end{aligned}\tag{37}$$

Case II: Either $\gamma_l\leq 0$ or $\omega_l\leq 0$ but not both. We can take $\nu_{1l}=1$ (if $\gamma_l\leq 0$) or $\nu_{2l}=1$ (if $\omega_l\leq 0$), it is easy to see that (37) still follows.

Case III: Both $\gamma_l \le 0$ and $\omega_l \le 0$. This is equivalent to

$$\frac{2m(q+1)+aq+b}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1-\frac{2m-2l}{N-1}$$

and

$$\frac{2m(p+1)+a+bp}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1-\frac{2l}{N-1},$$

which gives

$$\frac{2m(p+2+q)+a(q+1)+b(p+1)}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 2-\frac{2m}{N-1}.$$

Contradiction to $pq>1$ and $N \ge 2m+1$.

From (37) we obtain the following upper bound on $G_1(R)$.

$$\begin{aligned} G_1(R) \le C R^{N+2m} \sum_{l=0}^{m} \Big\{ &\left(\left\| D_x^{2m-2l} u_l \right\|_{1+\varepsilon} + R^{-2m+2l} \|u_l\|_1 \right)^{\nu_{1l}} \\ &\cdot \left(\left\| D_x^{2m-2l} u_l \right\|_k + R^{-2m+2l} \|u_l\|_1 \right)^{1-\nu_{1l}} \cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_{1+\varepsilon} + R^{-2l} \|v_{m-l}\|_1 \right)^{\nu_{2l}} \\ &\cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_d + R^{-2l} \|v_{m-l}\|_1 \right)^{1-\nu_{2l}} \Big\}. \end{aligned} \tag{38}$$

Step 2. *Estimation of $G_2(R)$ in terms of sutiable norms of $D_x^{2m} u(R)$ and $D_x^{2m} v(R)$.*

Fix $l \in \{0,1,2,\cdots,m-1\}$. For $\frac{1}{\beta_l}+\frac{1}{\beta_l'}=1$,

$$\begin{aligned} &\int_{S^{N-1}} \left(\left| u_{m-l-1}' \right| + R^{-1} |u_{m-l-1}| \right) \left(|v_l'| + R^{-1} |v_l| \right) \\ &\le \left(\left\| u_{m-l-1}' \right\|_{\beta_l} + R^{-1} \|u_{m-l-1}\|_{\beta_l} \right) \left(\|v_l'\|_{\beta_l'} + R^{-1} \|v_l\|_{\beta_l'} \right). \end{aligned} \tag{39}$$

By Lemma 2.5 and Hölder inequality,

$$\begin{aligned} R^{-1} \|u_{m-l-1}\|_{\beta_l} \le & C R^{-1} \left(\|D_\theta u_{m-l-1}\|_{\beta_l} + \|u_{m-l-1}\|_1 \right) \\ \le & C \left(\|D_x u_{m-l-1}\|_{\beta_l} + R^{-1} \|u_{m-l-1}\|_1 \right), \end{aligned} \tag{40}$$

$$R^{-1} \|v_l\|_{\beta_l'} \le C R^{-1} \left(\|D_\theta v_l\|_{\beta_l'} + \|v_l\|_1 \right) \le C \left(\|D_x v_l\|_{\beta_l'} + R^{-1} \|v_l\|_1 \right). \tag{41}$$

By Lemma 2.5 for $\frac{1}{\rho_l} = \frac{1}{k} - \frac{2l+1}{N-1}$

$$\begin{aligned} \|D_x u_{m-l-1}\|_{\rho_l} \le & C \left(\left\| D_\theta^{2l+1} D_x u_{m-l-1} \right\|_k + \|D_x u_{m-l-1}\|_1 \right) \\ \le & C \left(R^{2l+1} \left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_k + \|D_x u_{m-l-1}\|_1 \right), \end{aligned} \tag{42}$$

and for $\frac{1}{\sigma_l} = \frac{1}{d} - \frac{2m-2l-1}{N-1}$

$$\begin{aligned} \|D_x v_l\|_{\sigma_l} \le & C \left(\left\| D_\theta^{2m-2l-1} D_x v_l \right\|_d + \|D_x v_l\|_1 \right) \\ \le & C \left(R^{2m-2l-1} \left\| D_x^{2m-2l-1} D_x v_l \right\|_d + \|D_x v_l\|_1 \right). \end{aligned} \tag{43}$$

For $\eta_l = \frac{N-1}{N-2l-2}$, $\kappa_l = \frac{N-1}{N-2m+2l}$, Lemma 2.5 implies

$$\begin{aligned} \|D_x u_{m-l-1}\|_{\eta_l} \le & C \left(\left\| D_\theta^{2l+1} D_x u_{m-l-1} \right\|_{1+\varepsilon} + \|D_x u_{m-l-1}\|_1 \right) \\ \le & C \left(R^{2l+1} \left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_{1+\varepsilon} + \|D_x u_{m-l-1}\|_1 \right) \end{aligned}$$

and

$$\begin{aligned}\|D_x v_l\|_{\kappa_l} \le & C\left(\left\|D_\theta^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1\right)\\ \le & C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1\right).\end{aligned}$$

Assumption $\frac{N+a}{p+1}+\frac{N+b}{q+1} > N-2m$ implies $\frac{1}{\rho_l}+\frac{1}{\sigma_l} < 1$. Therefore we can pick $\beta_l = z_l \in (1,\infty)$ in (39) such that

$$\begin{aligned}\frac{1}{k}-\frac{2l+1}{N-1} \le & \frac{1}{z_l} \le 1-\frac{2l+1}{N-1},\\ \frac{1}{d}-\frac{2m-2l-1}{N-1} \le & 1-\frac{1}{z_l} \le 1-\frac{2m-2l-1}{N-1}.\end{aligned} \tag{44}$$

Case I: $\rho_l > 0$, $\sigma_l > 0$. Hölder's inequality gives

$$\begin{aligned}\|D_x u_{m-l-1}\|_{z_l} \le & \|D_x u_{m-l-1}\|_{\eta_l}^{\tau_{1l}} \|D_x u_{m-l-1}\|_{\rho_l}^{1-\tau_{1l}}\\ \le & C\left(R^{2l+1}\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + \|D_x u_{m-l-1}\|_1\right)^{\tau_{1l}}\\ & \cdot \left(R^{2l+1}\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + \|D_x u_{m-l-1}\|_1\right)^{1-\tau_{1l}}\\ = & CR^{2l+1}\left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1}\|D_x u_{m-l-1}\|_1\right)^{\tau_{1l}}\\ & \cdot \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + R^{-2l-1}\|D_x u_{m-l-1}\|_1\right)^{1-\tau_{1l}},\end{aligned} \tag{45}$$

where

$$\frac{1}{z_l} = \frac{\tau_{1l}}{\eta_l} + \frac{1-\tau_{1l}}{\rho_l},$$

and

$$\begin{aligned}\|D_x v_l\|_{z_l'} \le & \|D_x v_l\|_{\kappa_l}^{\tau_{2l}} \|D_x v_l\|_{\sigma_l}^{1-\tau_{2l}}\\ \le & C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1\right)^{\tau_{2l}}\\ & \cdot \left(R^{2m-2l-1}\left\|D_x^{2m-2l-1} D_x v_l\right\|_d + \|D_x v_l\|_1\right)^{1-\tau_{2l}}\\ = & CR^{2m-2l-1}\left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2m+2l+1}\|D_x v_l\|_1\right)^{\tau_{2l}}\\ & \cdot \left(\left\|D_x^{2m-2l-1} D_x v_l\right\|_d + R^{-2m+2l+1}\|D_x v_l\|_1\right)^{1-\tau_{2l}},\end{aligned} \tag{46}$$

with

$$1-\frac{1}{z_l} = \frac{1}{z_l'} = \frac{\tau_{2l}}{\kappa_l} + \frac{1-\tau_{2l}}{\sigma_l}.$$

Combining (40) , (41) , (42) , (43) , (45) , (46) we have

$$
\begin{aligned}
&\int_{S^{N-1}} \left(\left|u'_{m-l-1}\right| + R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v'_l\right| + R^{-1}\left|v_l\right|\right) \\
\leq & \left(\left\|u'_{m-l-1}\right\|_{z_l} + R^{-1}\left\|u_{m-l-1}\right\|_{z_l}\right)\left(\left\|v'_l\right\|_{z'_l} + R^{-1}\left\|v_l\right\|_{z'_l}\right) \\
\leq & C\left(\left\|D_x u_{m-l-1}\right\|_{z_l} + R^{-1}\left\|u_{m-l-1}\right\|_1\right)\left(\left\|D_x v_l\right\|_{z'_l} + R^{-1}\left\|v_l\right\|_1\right) \\
\leq & CR^{2m}\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}} \\
&\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}} \\
&\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|\right)^{\tau_{2l}} \\
&\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_d + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|\right)^{1-\tau_{2l}}. \qquad (47)
\end{aligned}
$$

Case II: $\sigma_l \leq 0$ or $\rho_l \leq 0$ but not both. We can take $\tau_{1l} = 1$ (if $\rho_l \leq 0$) or $\tau_{2l} = 1$ (if $\sigma_l \leq 0$), it is easy to see that (47) still holds.

Case III: Both $\sigma_l \leq 0$ and $\rho_l \leq 0$. This is equivalent to

$$\frac{2m(q+1)+aq+b}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1 - \frac{2m-2l-1}{N-1}$$

and

$$\frac{2m(p+1)+a+bp}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1 - \frac{2l+1}{N-1},$$

which gives

$$\frac{2m(p+2+q)+a(q+1)+b(q+1)}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 2 - \frac{2m}{N-1}.$$

Contradiction to $pq > 1$ and $N \geq 2m+1$.

It follows from (47) that

$$
\begin{aligned}
&G_2(R) \\
\leq & CR^N\sum_{l=0}^{m-1}\int_{S^{N-1}}\left(\left|u'_{m-l-1}\right| + R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v'_l\right| + R^{-1}\left|v_l\right|\right) \\
\leq & CR^{N+2m}\sum_{l=1}^{m-1} \\
&\left\{\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}}\right. \\
&\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}} \\
&\cdot\left(\left\|D_x^{2m-2l-1}D_x v_l\right\|_{1+\varepsilon} + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|_1\right)^{\tau_{2l}} \\
&\left.\cdot\left(\left\|D_x^{2m-2l-1}D_x v_l\right\|_d + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|_1\right)^{1-\tau_{2l}}\right\}. \qquad (48)
\end{aligned}
$$

Step 3. *Measure and Feedback argument.*

We first define the following set

$$\Gamma_0^1(R) = \left\{r \in (R, 2R) : \|v(r)\|_p^p > KR^{-2m-\alpha-a}\right\},$$

$$\Gamma_0^2(R) = \left\{r \in (R, 2R) : \|u(r)\|_q^q > KR^{-2m-\beta-b}\right\},$$

$$\Gamma_1(R)=\left\{r\in(R,2R):\left\|D_x^{2m}u(r)\right\|_k^k>KR^{-N}F(4R)\right\},$$
$$\Gamma_2(R)=\left\{r\in(R,2R):\left\|D_x^{2m}v(r)\right\|_d^d>KR^{-N}F(4R)\right\},$$
$$\Gamma_3(R)=\left\{r\in(R,2R):\left\|D_x^{2m}u\right\|_{1+\varepsilon}^{1+\varepsilon}>KR^{-2m-\alpha+a\epsilon}\right\},$$
$$\Gamma_4(R)=\left\{r\in(R,2R):\left\|D_x^{2m}v\right\|_{1+\varepsilon}^{1+\varepsilon}>KR^{-2m-\beta+b\epsilon}\right\}.$$

For fixed $l\in\{1,2,\cdots,m-1\}$

$$\Gamma_{5l}(R)=\left\{r\in(R,2R):\|u_{m-l-1}(r)\|_1>KR^{-\alpha-2(m-l-1)}\right\},$$
$$\Gamma_{6l}(R)=\left\{r\in(R,2R):\|v_l(r)\|_1>KR^{-\beta-2l}\right\},$$
$$\Gamma_{7l}(R)=\left\{r\in(R,2R):\|D_xu_{m-l-1}(r)\|_1>KR^{-\alpha-2(m-l-1)-1}\right\},$$
$$\Gamma_{8l}(R)=\left\{r\in(R,2R):\|D_xv_l(r)\|_1>KR^{-\beta-2l-1}\right\}.$$

From Lemma 2.2 we deduce

$$cR^{N-2m-\alpha}\geq\int_R^{2R}r^a\|v(r)\|_p^pr^{N-1}dr\geq\left|\Gamma_0^1(R)\right|KR^{-2m-\alpha}R^{N-1},$$

which implies

$$\left|\Gamma_0^1(R)\right|<\frac{1}{4m+8}R$$

for $K\gg1$. Similarly, we get

$$\left|\Gamma_0^2(R)\right|<\frac{1}{4m+8}R$$

for $K\gg1$.

To estimate $\Gamma_1(R)$, by (19) in Lemma 2.6,

$$\begin{aligned}CF(4R)&\geq\int_R^{2R}\left\|D_x^{2m}u\right\|_k^kr^{N-1}dr\geq|\Gamma_1(R)|KR^{-N}F(4R)R^{N-1}\\&=|\Gamma_1(R)|KR^{-1}F(4R),\end{aligned}$$

From which it follows that for $K\gg1$

$$|\Gamma_1(R)|<\frac{1}{4m+8}R.$$

Similarly we deduce from (20), (21) and (22) in Lemma 2.6 that

$$|\Gamma_2(R)|<\frac{1}{4m+8}R,\ \ |\Gamma_3(R)|<\frac{1}{4m+8}R,\ \ |\Gamma_4(R)|<\frac{1}{4m+8}R.$$

By (15) in Lemma 2.6,

$$CR^{N-\alpha-2(m-l-1)}\geq\int_0^{2R}\|u_{m-l-1}\|_1r^{N-1}dr\geq|\Gamma_{5l}(R)|KR^{-\alpha-2(m-l-1)}R^{N-1},$$

which gives

$$|\Gamma_{5l}(R)|<\frac{1}{4m+8}R$$

when $K\gg1$ and similarly (16), (17) and (18) implies

$$|\Gamma_{6l}(R)|<\frac{1}{4m+8}R,\ \ |\Gamma_{7l}(R)|<\frac{1}{4m+8}R,\ \ |\Gamma_{8l}(R)|<\frac{1}{4m+8}R$$

when $K \gg 1$. In particular, when $K \gg 1$,

$$\Gamma(R) = (R, 2R) \setminus \left\{ \cup_{j=1}^{2} \Gamma_0^j(R) \cup_{i=1}^{4} \Gamma_i(R) \cup_{l=1}^{m-1} \cup_{j=5}^{8} \Gamma_{jl}(R) \right\} \neq \emptyset.$$

Pick $\widetilde{R} \in \Gamma(R)$, by (38) together with the observation that

$$u_m = |x|^a v^p, \qquad v_m = |x|^b u^q,$$

we have

$$\begin{aligned}
G_1\left(\widetilde{R}\right) \leq & C\widetilde{R}^N \sum_{l=0}^{m} \int_{S^{N-1}} |u_l| \, |v_{m-l}| \\
\leq & C\widetilde{R}^{N+2m} \sum_{l=0}^{m} \Big\{ \left(\left\| D_x^{2m-2l} u_l \right\|_{1+\varepsilon} + \widetilde{R}^{-2m+2l} \left\| u_l \right\|_1 \right)^{\nu_{1l}} \\
& \cdot \left(\left\| D_x^{2m-2l} u_l \right\|_k + \widetilde{R}^{-2m+2l} \left\| u_l \right\|_1 \right)^{1-\nu_{1l}} \\
& \cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_{1+\varepsilon} + \widetilde{R}^{-2l} \left\| v_{m-l} \right\|_1 \right)^{\nu_{2l}} \\
& \cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_d + \widetilde{R}^{-2l} \left\| v_{m-l} \right\|_1 \right)^{1-\nu_{2l}} \Big\} \\
\leq & CR^{N+2m} \sum_{l=0}^{m} \{ R^{\frac{a\varepsilon - (2m+\alpha)}{1+\varepsilon}\nu_{1l}} \left(R^{-N} F(4R) \right)^{\frac{1-\nu_{1l}}{k}} \left(R^{\frac{b\varepsilon - (2m+\beta)}{1+\varepsilon}} + R^{-2m-\beta} \right)^{\nu_{2l}} \\
& \cdot \left(R^{-\frac{N}{d}} F(4R)^{\frac{1}{d}} + R^{-2m-\beta} \right)^{1-\nu_{2l}} \Big\} \\
\leq & CR^{-\widehat{a}} F^{\widehat{b}}(4R),
\end{aligned}$$

with

$$\begin{aligned}
\widehat{a} =& \widehat{a}_\varepsilon \\
=& \min_l \{ -N - 2m + \frac{2m+\alpha - a\varepsilon}{1+\varepsilon} \nu_{1l} + \frac{2m+\beta - b\varepsilon}{1+\varepsilon} \nu_{2l} + \frac{N}{k}(1-\nu_{1l}) + \frac{N}{d}(1-\nu_{2l}) \}, \\
\widehat{b} =& \max_l \frac{1}{k}(1-\nu_{1l}) + \frac{1}{d}(1-\nu_{2l}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& G_2\left(\widetilde{R}\right) \\
\leq & CR^{N+2m} \\
& \cdot \sum_{l=1}^{m-1} \Big\{ \left(\left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_{1+\varepsilon} + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{\tau_{1l}} \\
& \cdot \left(\left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_k + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{1-\tau_{1l}} \\
& \cdot \left(\left\| D_x^{2m-2l-1} D_x v_l \right\|_{1+\varepsilon} + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{\tau_{2l}} \\
& \cdot \left(\left\| D_x^{2m-2l-1} D_x v_l \right\|_d + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{1-\tau_{2l}} \Big\}
\end{aligned}$$

$$
\begin{aligned}
\leq & CR^{N+2m} \\
& \cdot \sum_{l=1}^{m-1} \Big\{ \Big(\left\| D_x^{2m} u \right\|_{1+\varepsilon} + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \Big)^{\tau_{1l}} \\
& \cdot \left(\left\| D_x^{2m} u \right\|_k + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{1-\tau_{1l}} \\
& \cdot \Big(\left\| D_x^{2m} v \right\|_{1+\varepsilon} + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \Big)^{\tau_{2l}} \\
& \cdot \left(\left\| D_x^{2m} v \right\|_d + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{1-\tau_{2l}} \Big\} \\
\leq & CR^{N+2m} \\
& \cdot \sum_{l=1}^{m-1} \Big\{ \Big(R^{\frac{a\varepsilon-(2m+\alpha)}{1+\varepsilon}} + R^{-2l-1} R^{-\alpha-2(m-l-1)-1} + R^{-2l-2} R^{-\alpha-2(m-l-1)} \Big)^{\tau_{1l}} \\
& \cdot \Big(R^{-N/k} F^{\frac{1}{k}} (4R) + R^{-2l-1} R^{-\alpha-2(m-l-1)-1} + R^{-2l-2} R^{-\alpha-2(m-l-1)} \Big)^{1-\tau_{1l}} \\
& \cdot \Big(R^{\frac{b\varepsilon-(2m+\beta)}{1+\varepsilon}} + R^{-2l-1} R^{-\beta-2(m-l-1)-1} + R^{-2l-2} R^{-\beta-2(m-l-1)} \Big)^{\tau_{2l}} \\
& \cdot \Big(R^{-N/d} F^{\frac{1}{d}} (4R) + R^{-2l-1} R^{-\beta-2(m-l-1)-1} + R^{-2l-2} R^{-\beta-2(m-l-1)} \Big)^{1-\tau_{2l}} \Big\} \\
\leq & CR^{-\overline{a}} F^{\overline{b}} (4R).
\end{aligned}
$$

Here

$$
\begin{aligned}
& \overline{a} \\
=& \overline{a}_\varepsilon \\
=& \min_l \{ -N-2m+\frac{2m+\alpha-a\varepsilon}{1+\varepsilon}\tau_{1l}+\frac{2m+\beta-b\varepsilon}{1+\varepsilon}\tau_{2l}+\frac{N}{k}\left(1-\tau_{1l}\right)+\frac{N}{d}\left(1-\tau_{2l}\right)\}
\end{aligned}
$$

and

$$\overline{b} = \max_l \frac{1}{k}\left(1-\tau_{1l}\right) + \frac{1}{d}\left(1-\tau_{2l}\right).$$

We claim that there exists a constant $M > 0$ and a sequence $R_i \to \infty$ such that

$$F(4R_i) \leq MF(R_i).$$

Otherwise for any $M > 0$, there exists R_M such that for $R \geq R_M$

$$F(4R) > MF(R).$$

Since (u, v) is bounded, we have $F(R) \leq CR^N$, $R > 0$. Thus

$$M^i F(R_M) \leq F\left(4^i R_M\right) \leq CR_M^N \left(4^N\right)^i.$$

Contradiction for i large if $M > 4^N$.

Assume we have shown $a = a_\varepsilon = \min\left(\widehat{a_\varepsilon}\ ,\overline{a_\varepsilon}\right) > 0$, $b = b_\varepsilon = \max\left(\widehat{b},\ \overline{b}\right) < 1$, we have

$$F(R_i) \leq CR^{-a} F^b (4R_i) \leq CM^b R^{-a} F^b (R_i),$$

which gives

$$F(R_i) \leq CR_i^{-\frac{a}{1-b}}.$$

Letting $i \to \infty$, we deduce that

$$\int_{\mathbb{R}^n} \left[|x|^a u^{q+1} + |x|^b v^{p+1} \right] dx = 0,$$

hence $u = v \equiv 0$, a contradiction.

Step 4. *If* $\alpha > N - 2m - 1$, *then* $\overline{b}, \widehat{b} < 1$ *and* $\overline{a_\varepsilon}, \widehat{a_\varepsilon} > 0$ *for* $\varepsilon \ll 1$.

First we show $\widehat{a_\varepsilon} > 0$, $\widehat{b} < 1$. Since $\nu_{1l} = \left(\frac{1}{\delta_l} - \frac{1}{\gamma_l}\right)^{-1}\left(\frac{1}{\alpha_l} - \frac{1}{\gamma_l}\right)$, $\nu_{2l} = \left(\frac{1}{\psi_l} - \frac{1}{\omega_l}\right)^{-1}\left(\frac{1}{\alpha_l'} - \frac{1}{\omega_l}\right)$, to show $\widehat{b} < 1$, we need to show for all l,

$$\begin{aligned} &\frac{1}{k}(1-\nu_{1l}) + \frac{1}{d}(1-\nu_{2l}) = \widetilde{p}\widehat{A}_{1l} + \widetilde{q}\widehat{A}_{2l} \\ =&\widetilde{p}\left(\frac{N-2m+2l-1}{N-1} - \frac{1}{\alpha_l}\right) + \widetilde{q}\left(\frac{1}{\alpha_l} - \frac{2l}{N-1}\right) < 1. \end{aligned} \tag{49}$$

Here

$$\widetilde{p} = \frac{2mp(q+1)+a+bp}{2m(q+1)+aq+b}, \ \widetilde{q} = \frac{2mq(p+1)+aq+b}{2m(p+1)+a+bp}.$$

It then follows that

$$k = \frac{\widetilde{p}+1}{\widetilde{p}}, \ d = \frac{\widetilde{q}+1}{\widetilde{q}}.$$

And

$$\alpha \geq \beta$$

implies

$$\widetilde{p} \geq \widetilde{q}.$$

We have

$$\begin{aligned} \frac{1}{\widetilde{q}+1} &= \frac{2m(p+1)+a+bp}{2m(p+1)(q+1)+a(q+1)+b(p+1)}. \\ \widehat{A}_{1l} =& \frac{1}{\widetilde{p}+1}(1-\nu_{1l}) = \left(\frac{N-2m+2l-1}{N-1} - \frac{1}{\alpha_l}\right), \\ \widehat{A}_{2l} =& \frac{1}{\widetilde{q}+1}(1-\nu_{2l}) = \left(\frac{1}{\alpha_l} - \frac{2l}{N-1}\right). \end{aligned}$$

(49) is equivalent to

$$\widetilde{p}\frac{N-2m+2l-1}{N-1} - \widetilde{q}\frac{2l}{N-1} - 1 < \frac{\widetilde{p}-\widetilde{q}}{\alpha_l}. \tag{50}$$

Recall α_l is chosen to satisfy (31). Such $\alpha_l \in (1, \infty)$ satisfying (31) and (50) exists provided

$$\max\left(\frac{1}{k} - \frac{2m-2l}{N-1}, \frac{2l}{N-1}\right) \leq \min\left(1 - \frac{2m-2l}{N-1}, \frac{1}{\widetilde{q}+1} + \frac{2l}{N-1}\right) \tag{51}$$

and

$$\widetilde{p}\frac{N-2m+2l-1}{N-1} - \widetilde{q}\frac{2l}{N-1} - 1 < (\widetilde{p}-\widetilde{q})\left(1 - \frac{2m-2l}{N-1}\right), \tag{52}$$

$$\widetilde{p}\frac{N-2m+2l-1}{N-1} - \widetilde{q}\frac{2l}{N-1} - 1 < (\widetilde{p}-\widetilde{q})\left(\frac{1}{\widetilde{q}+1} + \frac{2l}{N-1}\right). \tag{53}$$

(51) follows from the assumption that $N \geq 2m+1$ and

$$\frac{1+\frac{a}{N}}{p+1} + \frac{1+\frac{b}{N}}{q+1} > 1 - \frac{2m}{N} > 1 - \frac{2m}{N-1}.$$

(52) is equivalent to

$$\widetilde{q}\frac{N-1-2m}{N-1} < 1,$$

which follows from

$$\widetilde{q} \le \frac{\widetilde{p}\left(\widetilde{q}+1\right)}{\widetilde{p}+1} = 1 + \frac{2m}{\alpha} < 1 + \frac{2m}{N-2m-1} = \frac{N-1}{N-1-2m}.$$

And (53) can be rewritten as

$$\frac{N-2m-1}{N-1}\widetilde{p} < \frac{\widetilde{p}+1}{\widetilde{q}+1},$$

which is equivalent to

$$\alpha > N-2m-1.$$

Finally, since

$$(2m+\alpha)(k-1) = \beta, \quad (2m+\beta)(d-1) = \alpha,$$

we can write for each l

$$\begin{aligned}
\widehat{a_{0l}} =& -N-2m+(2m+\alpha)\nu_{1l}+(2m+\beta)\nu_{2l}+\frac{N}{k}(1-\nu_{1l})+\frac{N}{d}(1-\nu_{2l}) \\
=& 2m-N+\alpha+\beta+(N-2m-\alpha-\beta)\left(\widetilde{p}\widehat{A}_{1l}+\widetilde{q}\widehat{A}_{2l}\right) \\
=& (2m-N+\alpha+\beta)\left(1-\widetilde{p}\widehat{A}_{1l}-\widetilde{q}\widehat{A}_{2l}\right) \\
>& 0.
\end{aligned}$$

It then follows $\widehat{a_0} > 0$, thus $\widehat{a_\varepsilon} > 0$ for $\varepsilon \ll 1$.

Secondly we show $\overline{a_\varepsilon} > 0$, $\overline{b} < 1$. This can be shown in a similar way as $\widehat{a_\varepsilon}, \widehat{b}$. We write all details for readers' convenience. Since $\tau_{1l} = \left(\frac{1}{\eta_l} - \frac{1}{\rho_l}\right)^{-1}\left(\frac{1}{z_l} - \frac{1}{\rho_l}\right)$, $\tau_{2l} = \left(\frac{1}{\kappa_l} - \frac{1}{\sigma_l}\right)^{-1}\left(\frac{1}{z_l'} - \frac{1}{\sigma_l}\right)$, to show $\overline{b} < 1$, we need to show for all l,

$$\begin{aligned}
&\frac{1}{k}(1-\tau_{1l}) + \frac{1}{d}(1-\tau_{2l}) = \widetilde{p}A_{1l} + \widetilde{q}A_{2l} \\
=& \widetilde{p}\left(\frac{N-2l-2}{N-1} - \frac{1}{z_l}\right) + \widetilde{q}\left(\frac{1}{z_l} - \frac{2m-2l-1}{N-1}\right) < 1.
\end{aligned} \tag{54}$$

Here we used

$$\begin{aligned}
A_{1l} =& \frac{1}{\widetilde{p}+1}(1-\tau_{1l}) = \left(\frac{N-2l-2}{N-1} - \frac{1}{z_l}\right), \\
A_{2l} =& \frac{1}{\widetilde{q}+1}(1-\tau_{2l}) = \left(\frac{1}{z_l} - \frac{2m-2l-1}{N-1}\right).
\end{aligned}$$

(54) is equivalent to

$$\widetilde{p}\frac{N-2l-2}{N-1} - \widetilde{q}\frac{2m-2l-1}{N-1} - 1 < \frac{\widetilde{p}-\widetilde{q}}{z_l}. \tag{55}$$

Recall z_l is chosen to satisfy (44). Such $z_l \in (1,\infty)$ satisfying (44) and (55) exists provided

$$\max\left(\frac{\widetilde{p}}{\widetilde{p}+1} - \frac{2l+1}{N-1}, \frac{2m-2l-1}{N-1}\right) \le \min\left(1 - \frac{2l+1}{N-1}, \frac{1}{\widetilde{q}+1} + \frac{2m-2l-1}{N-1}\right) \tag{56}$$

and

$$\widetilde{p}\frac{N-2l-2}{N-1} - \widetilde{q}\frac{2m-2l-1}{N-1} - 1 < (\widetilde{p}-\widetilde{q})\left(1 - \frac{2l+1}{N-1}\right), \tag{57}$$

$$\widetilde{p}\frac{N-2l-2}{N-1}-\widetilde{q}\frac{2m-2l-1}{N-1}-1<(\widetilde{p}-\widetilde{q})\left(\frac{1}{q+1}+\frac{2m-2l-1}{N-1}\right). \tag{58}$$

(56) follows from

$$\frac{1+\frac{a}{N}}{p+1}+\frac{1+\frac{b}{N}}{q+1}>1-\frac{2m}{N}>1-\frac{2m}{N-1}.$$

(57) is equivalent to

$$\widetilde{q}\frac{N-1-2m}{N-1}<1,$$

which follows from

$$\widetilde{q}\leq\frac{\widetilde{p}\,(\widetilde{q}+1)}{\widetilde{p}+1}=1+\frac{2m}{\alpha}<1+\frac{2m}{N-2m-1}=\frac{N-1}{N-1-2m}.$$

And lastly (58) can be rewritten as

$$\frac{N-2m-1}{N-1}\widetilde{p}<\frac{\widetilde{p}+1}{\widetilde{q}+1},$$

which is equivalent to

$$\alpha>N-2m-1.$$

Finally, for each l

$$\begin{aligned}\overline{a_{0l}}=&-2m-N+(2m+\alpha)\,(1-(\widetilde{p}+1)\,A_{1l})\\&+(2m+\beta)\,(1-(\widetilde{q}+1)\,A_{2l})+N\widetilde{p}A_{1l}+N\widetilde{q}A_{2l}\\=&2m-N+\alpha+\beta+(N-2m-\alpha-\beta)\,(\widetilde{p}A_{1l}+\widetilde{q}A_{2l})\\=&(2m-N+\alpha+\beta)\,(1-\widetilde{p}A_{1l}-\widetilde{q}A_{2l})\\>&0.\end{aligned}$$

It then follows $\overline{a_0}>0$, thus $\overline{a_\varepsilon}>0$ for $\varepsilon\ll 1$.

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