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A LIOUVILLE-TYPE THEOREM FOR HIGHER ORDER ELLIPTIC SYSTEMS

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Abstract. We prove there are no positive solutions to higher order elliptic system

$$\begin{cases} (-\Delta)^m u = v^p \\ (-\Delta)^m v = u^q \end{cases} \text{in } \mathbb{R}^N$$

if $p \geq 1$, $q \geq 1$, and $(p,q) \neq (1,1)$ satisfies $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N}$ and $\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) > \frac{N-2m-1}{m}$. Moreover, if $N = 2m+1$ or $N = 2m+2$, this system admits no positive solutions if $p \geq 1$, $q \geq 1$, $(p,q) \neq (1,1)$ satisfies $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N}$.

1. **Introduction.** In this paper, we consider positive solutions ($u > 0,\ v > 0$) of the following higher order elliptic system

$$\begin{cases} (-\Delta)^m u = v^p \\ (-\Delta)^m v = u^q \end{cases} \text{in } \mathbb{R}^N, \tag{1}$$

where $p > 0$, $q > 0$ and $N \geq 3$. We are mainly concerned with the question of nonexistence of such positive solutions.

When $m = 1$, (1) becomes the Lane-Emden system

$$\begin{cases} \Delta u + v^p = 0 \\ \Delta v + u^q = 0 \end{cases} \text{in } \mathbb{R}^N. \tag{2}$$

It has been conjectured that the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ is the dividing curve for existence and nonexistence of positive solutions of (2) . The conjecture was completely solved in the case of radial solutions [5, 8, 10]. Mitidieri [5] showed that there is no positive radial solutions to (2) below the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ if $p > 1$, $q > 1$; the condition $p > 1$, $q > 1$ was later relaxed to $p > 0$, $q > 0$ by Serrin and Zou [8, 10]. Furthermore, it is proved by Serrin and Zou [10] that there are infinitely many positive radial solutions above the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. Therefore, $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ serves as the dividing curve for existence and nonexistence of positive radial solutions of (2) .

The question for the general positive solutions to (2), to the best of our knowledge, has not been completely solved yet. Partial answers have been given over

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the years. Souto [12] proved nonexistence of positive C^2 solutions below the curve $\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-2}{N-1}$ when p, $q>0$. Felmer and de Figureiredo [2] showed that when $0<p,q\leq\frac{N+2}{N-2}$ and $(p,q)\neq\left(\frac{N+2}{N-2},\frac{N+2}{N-2}\right)$, (2) has no positive C^2 solutions. Further evidence supporting the conjecture can be found in [6], where it is shown that there exists no positive supersolutions to (2) below the curve

$$\left\{p>0,q>0:\ \frac{1}{p+1}+\frac{1}{q+1}=1-\frac{2}{N-2}\max\left(\frac{1}{p+1},\frac{1}{q+1}\right)\right\}. \tag{3}$$

We refer to (3) as S curve, and the hyperbola in the conjecture $\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-2}{N}$ will be referred as Sobolev's hyperbola throughout the paper. For $0<p,q$, if $pq\leq 1$ or $pq>1$ and $\max\left(\frac{2(p+1)}{pq-1},\frac{2(q+1)}{pq-1}\right)\geq N-2$, nonexistence of positive solutions was proved by Serrin and Zou in [9]. Direct calculation shows this is the same range of (p,q) as region below and on the S curve. Furthermore, Serrin and Zou [9] showed (2) admits no positive solutions satisfying algebraic growth at infinity below the Sobolev hyperbola when $N=3$. For the special case $\min(p,q)=1$, the conjecture was proved by C.-S. Lin [3] . Busca and Manásevich [1] proved that if $p,q>0$, $pq>1$,

$$\frac{N-2}{2}\leq\min\left(\frac{2(p+1)}{pq-1},\frac{2(q+1)}{pq-1}\right)\leq\max\left(\frac{2(p+1)}{pq-1},\frac{2(q+1)}{pq-1}\right)<N-2,$$

and

$$\left(\frac{2(p+1)}{pq-1},\frac{2(q+1)}{pq-1}\right)\neq\left(\frac{N-2}{2},\frac{N-2}{2}\right),$$

then there is no positive classical solutions to (2) . Most recently, the conjecture was fully solved in the case $N=3$ by Poláčik, Quittner and Souplet [7] and by Souplet [11] when $N=4$. Souplet also proved the conjecture when $N\geq 5$ under the additional assumption that $\max\left(\frac{2(p+1)}{pq-1},\frac{2(q+1)}{pq-1}\right)>N-3$.

Comparing to the Lane-Emden system, less is known about the higher order system (1) . In the single equation case, Mitidieri [5] proved that for $1<q<\frac{N+4m}{N-4m}$, $N>4m$, the problem

$$\begin{cases}\Delta^{2m}u=u^q\\ (-\Delta)^s u\geq 0,\ s=1,2\cdots,2m-1\end{cases}$$

in $\mathbb{R}^N$ has no positive radial solution of class $C^{4m}(\mathbb{R}^N)$. For the system case, it is proved in [4, 14] that if $N\geq 3$, $N>2m$, if $p\geq 1$, $q\geq 1$, $(p,q)\neq(1,1)$ satisfying

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2m}{N},$$

then system (1) has no positive radial solutions. For general solutions, the results in [4, 14] show that if $p,q\geq 1$, $(p,q)\neq(1,1)$ satisfies

$$\max\left(\frac{2m(p+1)}{pq-1},\frac{2m(q+1)}{pq-1}\right)\geq N-2m,$$

then system (1) admits no positive solutions. It is also proved in [4] that system (1) does not admit any positive solutions if $1<p,q<\frac{N+2m}{N-2m}$. Under the additional assumptions $(-\Delta)^i u>0,(-\Delta)^i v>0$ for $i=1,2\cdots,m-1$, Yan [14] proved system (1) admits no positive solutions if $pq\leq 1$.

In this paper, we prove the following improved Liouville type theorem.

Theorem 1.1. *$N \geq 3$, $N > 2m$, if $p \geq 1$, $q \geq 1, (p,q) \neq (1,1)$ satisfies*

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N} \tag{4}$$

and

$$\max\left(\frac{2m\left(p+1\right)}{pq-1}, \frac{2m\left(q+1\right)}{pq-1}\right) > N - 2m - 1,$$

then the problem (1) has no positive solutions of class $C^{2m}\left(\mathbb{R}^N\right)$. Moreover, if $N = 2m+1$ or $2m+2$, $p \geq 1, q \geq 1$, and $(p,q) \neq (1,1)$ satisfies (4), then (1) admits no positive solutions.

The paper is organized as follows. Section 2 presents some technical Lemmas as preparation, and Section 3 is devoted to the proof of Theorem 1.1. Our proof relies on a Rellich-Pohozaev identity combined with an adapted idea of measure and feedback argument in Souplet's paper [11].

2. **Preparations.**

When $pq > 1$,we introduce the following notations

$$\alpha = \frac{2\left(p+1\right)}{pq-1}, \quad \beta = \frac{2\left(q+1\right)}{pq-1}$$

and assume $\alpha \geq \beta$ (i.e. $p \geq q$) throughout the rest of the paper. The assumption

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2m}{N}$$

can be rewritten as $m\alpha + m\beta > N - 2m$. For $w \in C\left(\mathbb{R}^N\right)$, we denote the sphearical average of w by

$$\overline{w}\left(r\right) = \frac{1}{\omega_N}\int_{S^{N-1}} w\left(r,\theta\right) ds, \quad r > 0,$$

where ω_N is the area of the unit sphere S^{N-1}.

We quote the following growth estimates from [14].

Lemma 2.1. *(Lemma 2.5 and Lemma 3.3 [14]) If $pq = 1$, there is no positive solution of (1). If (u,v) is a positive solution of (1)and $p,q \geq 1$, and $pq > 1$, there exists a positive constant $M = M\left(p,q,n\right)$ such that*

$$\overline{u}\left(r\right) \leq Mr^{-m\alpha}, \qquad \overline{v}\left(r\right) \leq Mr^{-m\beta} \qquad \text{for} \quad r > 0. \tag{5}$$

and for $k = 1, \cdots, m-1$, $u_k = \left(-\Delta\right)^k u, v_k = \left(-\Delta\right)^k v$, we have

$$\left(-\Delta\right)^i u > 0, \quad \left(-\Delta\right)^i v > 0,\ i = 1, 2, \cdots, m-1.$$

$$\overline{u_k}\left(r\right) \leq Mr^{-m\alpha-2k}, \qquad \overline{v_k}\left(r\right) \leq Mr^{-m\beta-2k} \qquad \text{for} \quad r > 0. \tag{6}$$

Lemma 2.2. *(Lemma 3.4 [14]) Suppose that $p,q \geq 1$ and (u,v) is a positive solution of (1). Then we have*

$$\int_{B_R} u^q \leq cR^{N-2m-m\beta}, \quad \int_{B_R} v^p \leq cR^{N-2m-m\alpha}, \tag{7}$$

where $c = c\left(p,q,n\right)$.

We state the following interpolation inequalities and elliptic estimates.

Lemma 2.3. *(L^p estimates on B_R) Given $1 < k < \infty$, $R > 0$, and $z \in W^{2m,k}(B_{2R})$, then*

$$\int_{B_R} \left|D^{2m} z\right|^k \leq C \left(\int_{B_{2R}} \left|\Delta^m z\right|^k + R^{-2mk} \int_{B_{2R}} |z|^k \right).$$

Proof. Lemma follows from standard elliptic L^p estimates for second order elliptic equations and interpolation inequalities. □

Lemma 2.4. *For any $R > 0$, $l = 1, 2, \cdots m-1$, we have*

$$\int_{B_R} |\nabla_x u_l| \leq CR \int_{B_{2R}} |u_{l+1}| + CR^{-1} \int_{B_{2R}} |u_l|.$$

Lemma 2.5. *(Sobolev inequality on S^{N-1}) Let $N \geq 2$, $j \geq 1$, $1 < \mu < \lambda \leq \infty$ be such that $\mu \neq \frac{N-1}{j}$, and $w \in W^{j,\lambda}(S^{N-1})$, then*

$$\|w\|_\lambda \leq C \left(\left\| D_\theta^j w \right\|_\mu + \|w\|_1 \right),$$

where

$$\begin{aligned} \frac{1}{\mu} - \frac{1}{\lambda} &= \frac{j}{N-1} \quad \text{if } \mu < \frac{N-1}{j}, \\ \lambda &= \infty \quad \text{if } \mu > \frac{N-1}{j}. \end{aligned}$$

Lemma 2.6. *For any $R > 0$, the following estimates hold for $l = 1, 2 \cdots, m-1$:*

$$\int_0^R \|u_l(r)\|_1 r^{N-1} dr \leq Cr^{N-m\alpha-2l}, \tag{8}$$

$$\int_0^R \|v_l(r)\|_1 r^{N-1} dr \leq Cr^{N-m\beta-2l}, \tag{9}$$

$$\int_0^R \|D_x u_l\|_1 r^{N-1} dr \leq Cr^{N-m\alpha-2l-1}, \tag{10}$$

$$\int_0^R \|D_x v_l\|_1 r^{N-1} dr \leq Cr^{N-m\beta-2l-1}, \tag{11}$$

$$\int_0^R \left\|D_x^{2m} u\right\|_k^k r^{N-1} dr \leq CF(2R), \tag{12}$$

$$\int_0^R \left\|D_x^{2m} v\right\|_d^d r^{N-1} dr \leq CF(2R), \tag{13}$$

$$\int_0^R \left\|D_x^{2m} u\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \leq CR^{N-2m-m\alpha}, \tag{14}$$

$$\int_0^R \left\|D_x^{2m} v\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \leq CR^{N-2m-m\beta}, \tag{15}$$

where

$$k = \frac{p+1}{p}, \quad d = \frac{q+1}{q}, \quad \text{and} \quad F(R) = \int_{B_R} \left[v^{p+1} + u^{q+1}\right] dx.$$

Proof. (8), (9) are restatements of Lemma 2.1. (10) and (11) follows directly from Lemma 2.1 and Lemma 2.4.

To prove (12), Lemma 2.3 implies

$$\begin{aligned}\int_0^R \left\|D_x^{2m}u\right\|_k^k r^{N-1}dr &= \int_{B_R}\left|D^{2m}u\right|^k \\ &\leq C\left(\int_{B_{2R}}\left|\Delta^m u\right|^k + R^{-2mk}\int_{B_{2R}}u^k\right) \\ &= C\int_{B_{2R}}v^{p+1} + R^{-2mk}\int_{B_{2R}}u^k\end{aligned}$$

By Hölder's inequality and the fact that $k=\frac{p+1}{p}<q+1$, $F(R)\geq F(1)>0$, $R\geq 1$, we obtain

$$\begin{aligned}R^{-2mk}\int_{B_{2R}}u^k &\leq CR^{-2mk}R^{N\frac{pq-1}{p(q+1)}}\left(\int_{B_{2R}}u^{q+1}\right)^{\frac{p+1}{p(q+1)}} \\ &\leq CR^{-\frac{\chi}{p}}F(2R)^{\frac{p+1}{p(q+1)}} \\ &\leq CR^{-\frac{\chi}{p}}F(2R)\left(F(1)\right)^{\frac{p+1}{p(q+1)}-1} \\ &\leq CR^{-\frac{\chi}{p}}F(2R),\end{aligned}$$

where

$$\chi = 2m(p+1) - N\frac{pq-1}{q+1}.$$

Since

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2m}{N},$$

we have $\chi>0$, and (12) follows. (13) is proved similarly.

Lastly we prove (14).

$$\begin{aligned}\int_0^R \left\|D_x^{2m}u\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1}dr &\leq C\left(\int_{B_{2R}}\left|\Delta^m u\right|^{1+\varepsilon} + R^{-2m(1+\varepsilon)}\int_{B_{2R}}u^{1+\varepsilon}\right) \\ &= C\left(\int_{B_{2R}}v^{p(1+\varepsilon)} + R^{-2m(1+\varepsilon)}\int_{B_{2R}}u^{1+\varepsilon}\right) \\ &\leq C\left(\int_{B_{2R}}v^{p} + R^{-2m(1+\varepsilon)}\int_{B_{2R}}u\right) \\ &\leq C\left(R^{N-mp\beta} + R^{-2m(1+\varepsilon)}\cdot R^{N-m\alpha}\right) \\ &\leq CR^{N-mp\beta}.\end{aligned}$$

Here we used the boundedness of u and v in the second inequality (the boundedness can be see from Theorem 3.1 in Section 3), and the fact $\alpha+2=p\beta$ in the last inequality. (15) is proved similarly using the boundedness of u, v and $\beta+2=q\alpha$. □

In the rest of the section, we prove a Rellich-Pohozaev identity. We recall the following function defined in [5]

$$R_n(u,v) = \int_\Omega \left[\Delta^n u\,(x,\nabla v) + \Delta^n v\,(x,\nabla u)\right]dx,$$

where $\Omega \subset \mathbb{R}^N$, $u, v \in C^{2n}\left(\overline{\Omega}\right), n \geq 1$. If $n = 1$, we have

$$\begin{aligned} R_1(u,v) &= \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial n}(x,\nabla v) + \frac{\partial v}{\partial n}(x,\nabla u) - (\nabla u, \nabla v)(x,n) \right\} ds \\ &\quad + (N-2)\int_{\Omega} (\nabla u, \nabla v)\, dx. \end{aligned}$$

If $n = 2$, we get

$$R_2(u,v) = R_1(\Delta u, v) + R_1(u, \Delta v) - B(u,v), \tag{16}$$

where

$$B(u,v) = \int_{\partial\Omega} \Delta u \Delta v\,(x,n)\, ds - N\int_{\Omega} \Delta u \Delta v dx. \tag{17}$$

We quote the following Lemma from [5].

Lemma 2.7. *(Lemma 2.2 in [5]) If $u, v \in C^{2n}\left(\overline{\Omega}\right)$, then for $1 \leq s \leq n-2$, we have*

$$R_n(u,v) = \sum_{l=0}^{s} R_{n-s}\left(\Delta^l u, \Delta^{s-l} v\right) - \sum_{l=0}^{s-1} R_{n-(s+1)}\left(\Delta^{l+1} u, \Delta^{s-l} v\right). \tag{18}$$

Remark 1. An immediate consequence of Lemma 2.7 is the following implicit form of Rellich's identity. If $u, v \in C^{2n}\left(\overline{\Omega}\right)$, then

$$R_n(u,v) = \sum_{l=0}^{n-1} R_1\left(\Delta^l u, \Delta^{n-1-l} v\right) - \sum_{l=0}^{n-2} B\left(\Delta^l u, \Delta^{n-2-l} v\right). \tag{19}$$

Proof. Choose $s = n-2$ in (18), taking into account of (16) and (17), then (19) follows. □

Let's write

$$\begin{aligned} u^{q+1}(r) &= \int_{S^{N-1}} u^{q+1}(r,\theta)\, d\theta, \\ v^{p+1}(r) &= \int_{S^{N-1}} v^{p+1}(r,\theta)\, d\theta, \end{aligned}$$

We have the following Rellich-Pohozav identity.

Lemma 2.8. *For any $a_1 + a_2 = N - 2m$, $r > 0$, we have*

$$\begin{aligned}
& \left(\frac{N}{p+1} - a_1\right) \int_{B_r} v^{p+1} dx + \left(\frac{N}{q+1} - a_2\right) \int_{B_r} u^{q+1} dx \\
= & \frac{1}{p+1} v^{p+1}(r) r^N + \frac{1}{q+1} u^{q+1}(r) r^N \\
& - (-1)^m \left\{ \sum_{l=0}^{m-1} 2r^N \int_{S^{N-1}} \frac{\partial \Delta^l u}{\partial n} \cdot \frac{\partial \Delta^{m-1-l} v}{\partial n} ds \right. \\
& - \sum_{l=0}^{m-1} r^N \int_{S^{N-1}} \left(\nabla \Delta^l u, \nabla \Delta^{m-1-l} v\right) ds \\
& - \sum_{l=0}^{m-2} r^N \int_{S^{N-1}} \left(\Delta^{l+1} u, \Delta^{m-1-l} v\right) ds \\
& + \sum_{l=0}^{m-1} (2m - 2l - 2 + a_1) r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^l u}{\partial n} \Delta^{m-1-l} v ds \\
& \left. + \sum_{l=0}^{m-1} (a_2 + 2l) r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^{m-1-l} v}{\partial n} \Delta^l u \; ds \right\}.
\end{aligned}$$

Proof. By (1), we have

$$\begin{aligned}
(-1)^m R_m(u,v) &= \int_{B_r} \left[(-\Delta)^m u(x)(x, \nabla v) + (-\Delta)^m v(x, \nabla u)\right] dx \\
&= \int_{B_r} \left[v^p(x)(x, \nabla v) + u^q(x)(x, \nabla u)\right] dx \\
&= \int_{\partial B_r} \left[\frac{v^{p+1}}{p+1}(x,n) + \frac{u^{q+1}}{q+1}(x,n)\right] ds \\
&\quad - \frac{N}{p+1} \int_{B_r} v^{p+1} dx - \frac{N}{q+1} \int_{B_r} u^{q+1} dx \\
&= \frac{1}{p+1} v^{p+1}(r) r^N + \frac{1}{q+1} u^{q+1}(r) r^N \\
&\quad - \frac{N}{p+1} \int_{B_r} v^{p+1} dx - \frac{N}{q+1} \int_{B_r} u^{q+1} dx.
\end{aligned}$$

By (19), we have

$$\begin{aligned}
R_m(u,v) &= \sum_{l=0}^{m-1} R_1\left(\Delta^l u, \Delta^{m-1-l} v\right) - \sum_{l=0}^{m-2} B\left(\Delta^l u, \Delta^{m-2-l} v\right) \\
&= \sum_{l=0}^{m-1} \int_{\partial B_r} \left[\frac{\partial \Delta^l u}{\partial n}\left(x, \nabla \Delta^{m-1-l} v\right) + \frac{\partial \Delta^{m-1-l} v}{\partial n}\left(x, \nabla \Delta^l u\right)\right] ds \\
&\quad - \sum_{l=0}^{m-1} \int_{\partial B_r} \left(\nabla \Delta^l u, \nabla \Delta^{m-1-l} v\right)(x,n) ds
\end{aligned}$$

$$
\begin{aligned}
&+(N-2)\sum_{l=0}^{m-1}\int_{B_r}\left(\nabla\Delta^l u,\nabla\Delta^{m-1-l}v\right)dx\\
&-\sum_{l=0}^{m-2}\int_{\partial B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)(x,n)\,ds\\
&+N\sum_{l=0}^{m-2}\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx. \qquad (20)
\end{aligned}
$$

Use integration by parts, we have

$$
\int_{B_r}\left(\nabla\Delta^l u,\nabla\Delta^{m-1-l}v\right)dx=\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v ds-\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx,
$$

Then we can rewrite (20) as

$$
\begin{aligned}
R_m(u,v) &= \sum_{l=0}^{m-1}\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\left(x,\nabla\Delta^{m-1-l}v\right)+\frac{\partial\Delta^{m-1-l}v}{\partial n}\left(x,\nabla\Delta^l u\right)\right]ds\\
&\quad-\sum_{l=0}^{m-1}\int_{\partial B_r}\left(\nabla\Delta^l u,\nabla\Delta^{m-1-l}v\right)(x,n)\,ds\\
&\quad-\sum_{l=0}^{m-2}\int_{\partial B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)(x,n)\,ds\\
&\quad+(N-2)\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v ds\\
&\quad-(N-2)\sum_{l=0}^{m-1}\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx\\
&\quad+N\sum_{l=0}^{m-2}\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx\\
&= \sum_{l=0}^{m-1}\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\left(x,\nabla\Delta^{m-1-l}v\right)+\frac{\partial\Delta^{m-1-l}v}{\partial n}\left(x,\nabla\Delta^l u\right)\right]ds\\
&\quad-\sum_{l=0}^{m-1}\int_{\partial B_r}\left(\nabla\Delta^l u,\nabla\Delta^{m-1-l}v\right)(x,n)\,ds\\
&\quad-\sum_{l=0}^{m-2}\int_{\partial B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)(x,n)\,ds\\
&\quad+(N-2)\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v ds\\
&\quad-N\int_{B_r}\left(\Delta^m u,v\right)dx+2\sum_{l=0}^{m-1}\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx. \qquad (21)
\end{aligned}
$$

Recall

$$\begin{aligned}&\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx-\int_{B_r}\left(\Delta^{l}u,\Delta^{m-l}v\right)dx\\ =\;&\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u\right]ds.\end{aligned}$$

It then follows

$$\begin{aligned}&\sum_{l=0}^{m-1}\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx\\ =\;& m\int_{B_r}(u,\Delta^m v)+\sum_{l=0}^{m-1}\sum_{i=0}^{l}\int_{\partial B_r}\frac{\partial\Delta^i u}{\partial n}\Delta^{m-1-i}v\;ds\\ &-\sum_{l=0}^{m-1}\sum_{i=0}^{l}\int_{\partial B_r}\frac{\partial\Delta^{m-1-i}v}{\partial n}\Delta^i u\;ds,\end{aligned}\tag{22}$$

and

$$\begin{aligned}\int_{B_r}(\Delta^m u,v)\,dx \;=\;& \int_{B_r}(u,\Delta^m v)+\sum_{l=0}^{m-1}\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v\right.\\ &\left.-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u\right]ds.\end{aligned}\tag{23}$$

From (22) and (23) we deduce

$$\begin{aligned}&(N-2)\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v ds-N\int_{B_r}(\Delta^m u,v)\,dx\\ &+2\sum_{l=0}^{m-1}\int_{B_r}\left(\Delta^{l+1}u,\Delta^{m-1-l}v\right)dx\\ =\;&(2m-N)\int_{B_r}(u,\Delta^m v)\\ &+2\sum_{l=0}^{m-1}\sum_{i=0}^{l}\int_{\partial B_r}\left[\frac{\partial\Delta^i u}{\partial n}\Delta^{m-1-i}v-\frac{\partial\Delta^{m-1-i}v}{\partial n}\Delta^i u\right]ds\\ &+(N-2)\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v ds\\ &-N\sum_{l=0}^{m-1}\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u\right]ds\\ =\;&(2m-N)\int_{B_r}(u,\Delta^m v)\end{aligned}$$

$$
\begin{aligned}
& +2\sum_{l=0}^{m-1}(m-l)\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u\right]ds\\
& -2\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}vds+N\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l uds\\
= & -a_1\int_{B_r}(\Delta^m u,v)-a_2\int_{B_r}(u,\Delta^m v)\\
& +a_1\sum_{l=0}^{m-1}\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u\right]ds\\
& +2\sum_{l=0}^{m-1}(m-l)\int_{\partial B_r}\left[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l\right]uds\\
& -2\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}vds+N\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l uds\\
= & -a_1\int_{B_r}(\Delta^m u,v)-a_2\int_{B_r}(u,\Delta^m v)\\
& +\sum_{l=0}^{m-1}(2m-2l-2+a_1)\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}vds\\
& +\sum_{l=0}^{m-2}(N-2m+2l-a_1)\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l uds\\
= & -a_1\int_{B_r}(\Delta^m u,v)-a_2\int_{B_r}(u,\Delta^m v)\\
& +\sum_{l=0}^{m-1}(2m-2l-2+a_1)\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}vds\\
& +\sum_{l=0}^{m-1}(a_2+2l)\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l uds. \qquad (24)
\end{aligned}
$$

Hence, the conclusion follows from (21) and (24) . □

3. Proof of the theorem.

3.1. Bounded solution.

In this subsection, we prove that if the system does not admit bounded positive solutions, then it does not admit classical positive solutions. More precisely, we prove the following Theorems regarding bounded solutions.

Theorem 3.1. *Let $N \geq 3$, $p > 1$, $q > 1$ be fixed, and assume (1) does not admit any bounded nontrivial (nonnegative) solutions in $\mathbb{R}^N$, then it does not admit any nontrivial (nonnegative) solutions in $\mathbb{R}^N$, bounded or not. In particular, the conclusion holds if* $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) \geq N-2m$ *or if* $1 < p, q < \frac{N+2m}{N-2m}$.

Theorem 3.2. *Let $p, q > 1$. Assume (1) does not admit any bounded nontrival (nonnegative) solutions in $\mathbb{R}^N$. Let $\Omega \neq \mathbb{R}^N$ be a domain of $\mathbb{R}^N$. Then there exists $C = C(N, p, q, m) > 0$ (independent of Ω, u and v) such that any (nonnegative)* *solutions (u,v) of (1) in Ω satisfies*

$$u(x) \le C\,dist^{-m\alpha}(x,\partial\Omega), \quad x \in \Omega,$$

and

$$v(x) \le C\,dist^{-m\beta}(x,\partial\Omega), \quad x \in \Omega.$$

If Ω is an exterior domain, that is $\Omega \supset \{x \in \mathbb{R}^N : |x| > R\}$ for some $R > 0$, then it follows that

$$u(x) \le C\,|x|^{-m\alpha}, \;\; |x| \ge 2R,$$

and

$$v(x) \le C\,|x|^{-m\beta}, \;\; |x| \ge 2R.$$

In particular, the above conclusions hold if $\max\left(\dfrac{2m(p+1)}{pq-1}, \dfrac{2m(q+1)}{pq-1}\right) \ge N - 2m$ *or if* $1 < p, q < \dfrac{N+2m}{N-2m}$.

Proofs of Theorems 3.1 and 3.2 use idea of [7] in the case of $m = 1$, which relies on the following Doubling property Lemma and remark.

Lemma 3.3. *(Lemma 5.1 [7])Let (X,d) be a complete metric space, and let $\emptyset \neq D \subset \Sigma \subset X$ with Σ closed. Set $\Gamma = \Sigma \backslash D$. Finally, let $M : D \to (0,\infty)$ be bounded on compact subsets of D, and fix a real number $l > 0$. If $y \in D$ is such that*

$$M(y)\,dist(y,\Gamma) > 2l,$$

then there exists $x \in D$ such that

$$M(x)\,dist(x,\Gamma) > 2l, \quad M(x) \ge M(y),$$

and

$$M(z) \le 2M(x) \quad \text{for all } z \in D \cap \overline{B_X}(x, \delta M^{-1}(x)).$$

Remark 2. (Remark 5.2 [7]).

- **(a):** If $\Gamma = \emptyset$, then $\mathrm{dist}(x,\Gamma) = \infty$.
- **(b):** Take $X = \mathbb{R}^n$, take Ω an open subset of $\mathbb{R}^n$, put $D = \Omega$, $\Sigma = \overline{D}$; hence $\Gamma = \partial\Omega$. Then we have $\overline{B}(x, lM^{-1}(x)) \subset D$. Indeed, since D is open, implies $\mathrm{dist}(x, D^c) = \mathrm{dist}(x,\Gamma) > 2lM^{-1}(x)$.

Proof of Theorem 3.2. Assume the theorem fails. Then there exist sequences Ω_l, (u_l, v_k), $y_l \in \Omega_l$ such that (u_l, v_l) solves (1) on Ω_l and

$$M_l := u_l^{\frac{1}{m\alpha}} + v_l^{\frac{1}{m\beta}}, \quad l = 1, 2, \cdots$$

satisfies

$$M_l(y_l) > 2l\mathrm{dist}^{-1}(y_l, \partial\Omega_l).$$

By Lemma 3.3 and Remark 2, it follows that there exists $x_l \in \Omega_l$ such that

$$M_l(x_l) > 2l\mathrm{dist}^{-1}(x_l, \partial\Omega_l)$$

and

$$M_l(z) \le 2M_l(x_l), \quad |z - x_l| \le lM_l^{-1}(x_l).$$

Define rescaling of (u_l, v_l) as follows

$$\begin{aligned} \lambda_l &= M_l^{-1}(x_l) \\ \widetilde{u}_l(y) &= \lambda_l^{m\alpha} u_l(x_l + \lambda_l y), \;\; \widetilde{v}_l(y) = \lambda_l^{m\beta} v_l(x_l + \lambda_l y), \quad |y| \le l. \end{aligned}$$

Since $\alpha+2=p\beta$, $\beta+2=q\alpha$, $(\widetilde{u}_l,\widetilde{v}_l)$ satisfies

$$\begin{aligned}(-\Delta_y)^m \widetilde{u}_l(y) &= \widetilde{v}_l^p(y)\\ (-\Delta_y)^m \widetilde{v}_l(y) &= \widetilde{u}_l^q(y)\end{aligned}$$

for $|y|\le l$. Moreover,

$$\widetilde{u}_l^{\frac{1}{m\alpha}}(0)+\widetilde{v}_l^{\frac{1}{m\beta}}(0)=1$$

and

$$\widetilde{u}_l^{\frac{1}{m\alpha}}(y)+\widetilde{v}_l^{\frac{1}{m\beta}}(y)\le 2,\quad |y|\le l.$$

By standard elliptic L_p estimates and Sobolev embeddings, we conclude that subject to a subsequence, $(\widetilde{u}_l,\widetilde{v}_l)$ converges in $C_{loc}^{2m}(\mathbb{R}^N)$ to a (classical) solution $(\widetilde{u},\widetilde{v})$ of (1) in $\mathbb{R}^n$. Moreover, $\widetilde{u}^{\frac{1}{m\alpha}}(0)+\widetilde{v}^{\frac{1}{m\beta}}(0)=1$ and $\widetilde{u}^{\frac{1}{m\alpha}}(y)+\widetilde{v}^{\frac{1}{m\beta}}(y)\le 2$. i.e. $(\widetilde{u},\widetilde{v})$ is nontrivial and bounded, contradicts to the assumptions of the theorem. In particular, if $\max\left(\frac{2m(p+1)}{pq-1},\frac{2m(q+1)}{pq-1}\right)\ge N-2m$ or if $1<p,q<\frac{N+2m}{N-2m}$, Liouville theorems in [4] and [14] implies the assumptions in the theorem hold. □

Proof of Theorem 3.1. Assume (u,v) is a solution of (1) on $\mathbb{R}^N$ (bounded or not). Then for each $x_0\in\mathbb{R}^N$ and $R>0$, by applying Theorem 3.2 in $\Omega=B(x_0,R)$, we obtain

$$u(x_0)\le CR^{-m\alpha},\quad v(x_0)\le CR^{-m\beta}.$$

Letting $R\to\infty$, we obtain

$$u(x_0)=v(x_0)=0,$$

therefore

$$u\equiv v\equiv 0.$$

□

3.2. **Proof of theorem 1.1.** We shall adapt Souplet's idea [11] of a measure and feedback argument combined with Rellich-Pohazaev identity. Lemma 2.8 implies

$$F(R)\le CG_1(R)+CG_2(R),$$

where

$$G_1(R)=R^N\sum_{l=0}^{m}\int_{S^{N-1}}|u_l|\,|v_{m-l}|\,ds$$

and

$$G_2(R)=R^N\int_{S^{N-1}}\sum_{l=0}^{m-1}\left(\left|u'_{m-l-1}\right|+R^{-1}|u_{m-l-1}|\right)\left(|v'_l|+R^{-1}|v_l|\right)ds.$$

Following Souplet's idea, we shall prove there exist constants C, $a>0$, $b<1$ such that

$$F(R)\le CR^{-a}F^b(R). \tag{25}$$

It then follows

$$F(R)\to 0 \text{ as } R\to\infty,$$

which implies

$$u=v\equiv 0.$$

To prove (25), we follow a similar procedure as [11]. We shall first estimate $G_1(R)$ and $G_2(R)$ in terms of highest derivatives of the solution (u,v) and (u,v) in suitable L_p spaces. Then use a feedback and measure argument to evaluate those bounds in terms of $F(R)$.

Step 1. Estimation of $G_1(R)$ in terms of suitable norms of $D_x^{2m}u(R)$ and $D_x^{2m}v(R)$.

Fix $l \in \{0, 1, \cdots m\}$, Hölder's inequality gives

$$\begin{aligned} &\int_{S^{N-1}} |u_l| \, |v_{m-l}| \, ds \\ \leq \; & \|u_l\|_{\alpha_l} \|v_{m-l}\|_{\alpha_l'}, \end{aligned}$$

where $\frac{1}{\alpha_l} + \frac{1}{\alpha_l'} = 1$ is chosen so that

$$\begin{aligned} \frac{p}{p+1} - \frac{2m-2l}{N-1} &\leq \frac{1}{\alpha_l} \leq 1 - \frac{2m-2l}{N-1}, \\ \frac{q}{q+1} - \frac{2l}{N-1} &\leq 1 - \frac{1}{\alpha_l} \leq 1 - \frac{2l}{N-1}. \end{aligned} \tag{26}$$

Such α_l exists since by assumption,

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N} > 1 - \frac{2m}{N-1}.$$

Let

$$\begin{aligned} \frac{1}{\gamma_l} &= \frac{p}{p+1} - \frac{2m-2l}{N-1}, \frac{1}{\delta_l} = \frac{N-2m+2l-1}{N-1}, \\ \frac{1}{\omega_l} &= \frac{q}{q+1} - \frac{2l}{N-1}, \frac{1}{\psi_l} = \frac{N-2l-1}{N-1}. \end{aligned}$$

Case I. $\gamma_l > 0$, $\omega_l > 0$.

By Hölder's inequality, we have

$$\begin{aligned} \|u_l\|_{\alpha_l} &\leq \|u_l\|_{\delta_l}^{\nu_{1l}} \|u_l\|_{\gamma_l}^{1-\nu_{1l}}, \\ \|v_{m-l}\|_{\alpha_l'} &\leq \|v_{m-l}\|_{\psi_l}^{\nu_{2l}} \|v_{m-l}\|_{\omega_l}^{1-\nu_{2l}}, \end{aligned} \tag{27}$$

with

$$\begin{aligned} \frac{1}{\alpha_l} &= \frac{\nu_{1l}}{\delta_l} + \frac{1-\nu_{1l}}{\gamma_l}, \\ \frac{1}{\alpha_l'} &= \frac{\nu_{2l}}{\psi_l} + \frac{1-\nu_{2l}}{\omega_l}. \end{aligned}$$

Applying Lemma 2.5, we deduce

$$\begin{aligned} \|u_l\|_{\delta_l} &\leq C\left(\left\|D_\theta^{2m-2l}u_l\right\|_{1+\varepsilon} + \|u_l\|_1\right) \\ &\leq C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon} + \|u_l\|_1\right), \end{aligned} \tag{28}$$

$$\begin{aligned} \|u_l\|_{\gamma_l} &\leq C\left(\left\|D_\theta^{2m-2l}u_l\right\|_k + \|u_l\|_1\right) \\ &\leq C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_k + \|u_l\|_1\right), \end{aligned} \tag{29}$$

and

$$\begin{aligned} \|v_{m-l}\|_{\psi_l} &\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_{1+\varepsilon} + \|v_{m-l}\|_1\right) \\ &\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon} + \|v_{m-l}\|_1\right), \end{aligned} \tag{30}$$

$$\begin{aligned} \|v_{m-l}\|_{\omega_l} &\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_d + \|v_{m-l}\|_1\right) \\ &\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_d + \|v_{m-l}\|_1\right). \end{aligned} \tag{31}$$

Combining (27), (28), (29), (30) and (31), we conclude

$$\begin{aligned}
&\int_{S^{N-1}} |u_l|\,|v_{m-l}|\,ds \\
\leq\ & \|u_l\|_{\alpha_l}\,\|v_{m-l}\|_{\alpha_l'} \\
\leq\ & CR^{2m}\left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon}+R^{-2m+2l}\|u_l\|_1\right)^{\nu_{1l}} \\
&\cdot\left(\left\|D_x^{2m-2l}u_l\right\|_k+R^{-2m+2l}\|u_l\|_1\right)^{1-\nu_{1l}} \\
&\cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon}+R^{-2l}\|v_{m-l}\|_1\right)^{\nu_{2l}} \\
&\cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_d+R^{-2l}\|v_{m-l}\|_1\right)^{1-\nu_{2l}}. \qquad (32)
\end{aligned}$$

Case II. Either $\gamma_l \leq 0$ or $\omega_l \leq 0$ but not both. We can take $\nu_{1l} = 1$ (if $\gamma_l \leq 0$) or $\nu_{2l} = 1$ (if $\omega_l \leq 0$), it is easy to see that (32) still follows.
Case III. Both $\gamma_l \leq 0$ and $\omega_l \leq 0$. This is equivalent to

$$\frac{1}{p+1} > 1 - \frac{2m-2l}{N-1}$$

and

$$\frac{1}{q+1} > 1 - \frac{2l}{N-1},$$

which gives

$$\frac{1}{p+1} + \frac{1}{q+1} > 2 - \frac{2m}{N-1}.$$

Contradiction to $p \geq 1$, $q \geq 1$ and $N \geq 2m+1$.

From (32) we obtain the following upper bound on $G_1(R)$.

$$\begin{aligned}
G_1(R) \leq\ & CR^{N+2m}\sum_{l=0}^{m}\Big\{\left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon}+R^{-2m+2l}\|u_l\|_1\right)^{\nu_{1l}} \\
&\cdot\left(\left\|D_x^{2m-2l}u_l\right\|_k+R^{-2m+2l}\|u_l\|_1\right)^{1-\nu_{1l}} \\
&\cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon}+R^{-2l}\|v_{m-l}\|_1\right)^{\nu_{2l}} \\
&\cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_d+R^{-2l}\|v_{m-l}\|_1\right)^{1-\nu_{2l}}\Big\}. \qquad (33)
\end{aligned}$$

Step 2. Estimation of $G_2(R)$ in terms of sutiable norms of $D_x^{2m}u(R)$ and $D_x^{2m}v(R)$.

Fix $l \in \{0, 1, 2, \cdots, m-1\}$. For $\frac{1}{\beta_l} + \frac{1}{\beta_l'} = 1$,

$$\begin{aligned}
&\int_{S^{N-1}}\left(\left|u_{m-l-1}'\right|+R^{-1}|u_{m-l-1}|\right)\left(|v_l'|+R^{-1}|v_l|\right) \\
\leq\ & \left(\left\|u_{m-l-1}'\right\|_{\beta_l}+R^{-1}\|u_{m-l-1}\|_{\beta_l}\right)\left(\|v_l'\|_{\beta_l'}+R^{-1}\|v_l\|_{\beta_l'}\right). \qquad (34)
\end{aligned}$$

By Lemma 2.5,

$$\begin{aligned}
R^{-1}\|u_{m-l-1}\|_{\beta_l} \leq\ & CR^{-1}\left(\|D_\theta u_{m-l-1}\|_{\beta_l}+\|u_{m-l-1}\|_1\right) \\
\leq\ & C\left(\|D_x u_{m-l-1}\|_{\beta_l}+R^{-1}\|u_{m-l-1}\|_1\right), \qquad (35)
\end{aligned}$$

$$\begin{aligned} R^{-1}\|v_l\|_{\beta_l'} &\le CR^{-1}\left(\|D_\theta v_l\|_{\beta_l'}+\|v_l\|_1\right) \\ &\le C\left(\|D_x v_l\|_{\beta_l'}+R^{-1}\|v_l\|_1\right). \end{aligned} \tag{36}$$

By Lemma 2.5 for $\frac{1}{\rho_l}=\frac{p}{p+1}-\frac{2l+1}{N-1}$

$$\begin{aligned} \|D_x u_{m-l-1}\|_{\rho_l} &\le C\left(\left\|D_\theta^{2l+1}D_x u_{m-l-1}\right\|_k+\|D_x u_{m-l-1}\|_1\right) \\ &\le C\left(R^{2l+1}\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k+\|D_x u_{m-l-1}\|_1\right), \end{aligned} \tag{37}$$

and for $\frac{1}{\sigma_l}=\frac{q}{q+1}-\frac{2m-2l-1}{N-1}$

$$\begin{aligned} \|D_x v_l\|_{\sigma_l} &\le C\left(\left\|D_\theta^{2m-2l-1}D_x v_l\right\|_d+\|D_x v_l\|_1\right) \\ &\le C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1}D_x v_l\right\|_d+\|D_x v_l\|_1\right). \end{aligned} \tag{38}$$

For $\eta_l=\frac{N-1}{N-2l-2}$, $\kappa_l=\frac{N-1}{N-2m+2l}$, Lemma 2.5 implies

$$\begin{aligned} \|D_x u_{m-l-1}\|_{\eta_l} &\le C\left(\left\|D_\theta^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+\|D_x u_{m-l-1}\|_1\right) \\ &\le C\left(R^{2l+1}\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+\|D_x u_{m-l-1}\|_1\right) \end{aligned}$$

and

$$\begin{aligned} \|D_x v_l\|_{\kappa_l} &\le C\left(\left\|D_\theta^{2m-2l-1}D_x v_l\right\|_{1+\varepsilon}+\|D_x v_l\|_1\right) \\ &\le C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1}D_x v_l\right\|_{1+\varepsilon}+\|D_x v_l\|_1\right). \end{aligned}$$

Assumption $\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2m}{N}>1-\frac{2m}{N-1}$ implies $\frac{1}{\rho_l}+\frac{1}{\sigma_l}<1$. Therefore we can pick $\beta_l=z_l\in(1,\infty)$ in (34) such that

$$\begin{aligned} \frac{p}{p+1}-\frac{2l+1}{N-1} &\le \frac{1}{z_l}\le 1-\frac{2l+1}{N-1}, \\ \frac{q}{q+1}-\frac{2m-2l-1}{N-1} &\le 1-\frac{1}{z_l}\le 1-\frac{2m-2l-1}{N-1}. \end{aligned} \tag{39}$$

Case I. $\rho_l>0$, $\sigma_l>0$. Hölder's inequality gives

$$\begin{aligned} \|D_x u_{m-l-1}\|_{z_l} &\le \|D_x u_{m-l-1}\|_{\eta_l}^{\tau_{1l}}\|D_x u_{m-l-1}\|_{\rho_l}^{1-\tau_{1l}} \\ &\le C\left(R^{2l+1}\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+\|D_x u_{m-l-1}\|_1\right)^{\tau_{1l}} \\ &\quad\cdot\left(R^{2l+1}\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k+\|D_x u_{m-l-1}\|_1\right)^{1-\tau_{1l}} \\ &= CR^{2l+1}\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2l-1}\|D_x u_{m-l-1}\|_1\right)^{\tau_{1l}} \\ &\quad\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k+R^{-2l-1}\|D_x u_{m-l-1}\|_1\right)^{1-\tau_{1l}}, \end{aligned} \tag{40}$$

where

$$\frac{1}{z_l}=\frac{\tau_{1l}}{\eta_l}+\frac{1-\tau_{1l}}{\rho_l},$$

and

$$\begin{aligned}\|D_x v_l\|_{z_l'} &\leq \|D_x v_l\|_{\kappa_l}^{\tau_{2l}} \|D_x v_l\|_{\sigma_l}^{1-\tau_{2l}} \\ &\leq C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1}D_x v_l\right\|_{1+\varepsilon}+\|D_x v_l\|_1\right)^{\tau_{2l}} \\ &\quad \cdot\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1}D_x v_l\right\|_d+\|D_x v_l\|_1\right)^{1-\tau_{2l}} \\ &= CR^{2m-2l-1}\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2m+2l+1}\|D_x v_l\|_1\right)^{\tau_{2l}} \\ &\quad \cdot\left(\left\|D_x^{2m-2l-1}D_x v_l\right\|_d+R^{-2m+2l+1}\|D_x v_l\|_1\right)^{1-\tau_{2l}}, \qquad (41)\end{aligned}$$

with

$$1-\frac{1}{z_l}=\frac{1}{z_l'}=\frac{\tau_{2l}}{\kappa_l}+\frac{1-\tau_{2l}}{\sigma_l}.$$

Combining (35) , (36) , (37) , (38) , (40) , (41) we have

$$\begin{aligned}&\int_{S^{N-1}}\left(\left|u_{m-l-1}'\right|+R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v_l'\right|+R^{-1}\left|v_l\right|\right) \\ \leq&\left(\left\|u_{m-l-1}'\right\|_{z_l}+R^{-1}\|u_{m-l-1}\|_{z_l}\right)\left(\|v_l'\|_{z_l'}+R^{-1}\|v_l\|_{z_l'}\right) \\ \leq& C\left(\|D_x u_{m-l-1}\|_{z_l}+R^{-1}\|u_{m-l-1}\|_1\right)\left(\|D_x v_l\|_{z_l'}+R^{-1}\|v_l\|_1\right) \\ \leq& CR^{2m}\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2l-1}\|D_x u_{m-l-1}\|_1+R^{-2l-2}\|u_{m-l-1}\|_1\right)^{\tau_{1l}} \\ &\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k+R^{-2l-1}\|D_x u_{m-l-1}\|_1+R^{-2l-2}\|u_{m-l-1}\|_1\right)^{1-\tau_{1l}} \\ &\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2m+2l+1}\|D_x v_l\|_1+R^{-2m+2l}\|v_l\|\right)^{\tau_{2l}} \\ &\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_d+R^{-2m+2l+1}\|D_x v_l\|_1+R^{-2m+2l}\|v_l\|\right)^{1-\tau_{2l}}. \qquad (42)\end{aligned}$$

Case II. $\sigma_l \leq 0$ or $\rho_l \leq 0$ but not both. We can take $\tau_{1l}=1$ (if $\rho_l \leq 0$) or $\tau_{2l}=1$ (if $\sigma_l \leq 0$), it is easy to see that (42) still holds.
Case III. Both $\sigma_l \leq 0$ and $\rho_l \leq 0$. This is equivalent to

$$\frac{1}{p+1}>1-\frac{2m-2l-1}{N-1}$$

and

$$\frac{1}{q+1}>1-\frac{2l+1}{N-1},$$

which gives

$$\frac{1}{p+1}+\frac{1}{q+1}>2-\frac{2m}{N-1}.$$

Contradiction to $p \geq 1$, $q \geq 1$ and $N \geq 2m+1$.

It follows from (42) that

$$\begin{aligned}
& G_2(R) \\
\leq\ & CR^N \sum_{l=0}^{m-1} \int_{S^{N-1}} \left(\left|u'_{m-l-1}\right| + R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v'_l\right| + R^{-1}\left|v_l\right|\right) \\
\leq\ & CR^{N+2m} \sum_{l=1}^{m-1} \Big\{ \Big(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 \\
& + R^{-2l-2}\left\|u_{m-l-1}\right\|_1 \Big)^{\tau_{1l}} \\
& \cdot \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1 \right)^{1-\tau_{1l}} \\
& \cdot \Big(\left\|D_x^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|_1 \Big)^{\tau_{2l}} \\
& \cdot \left(\left\|D_x^{2m-2l-1} D_x v_l\right\|_d + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|_1 \right)^{1-\tau_{2l}} \Big\}. \qquad (43)
\end{aligned}$$

Step 3. Measure and Feedback argument.

We first define the following set

$$\Gamma_0^1(R) = \left\{ r \in (R, 2R) : \|v(r)\|_p^p > KR^{-mp\beta} \right\},$$

$$\Gamma_0^2(R) = \left\{ r \in (R, 2R) : \|u(r)\|_q^q > KR^{-mq\alpha} \right\},$$

$$\Gamma_1(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} u(r)\right\|_k^k > KR^{-N} F(4R) \right\},$$

$$\Gamma_2(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} v(r)\right\|_d^d > KR^{-N} F(4R) \right\},$$

$$\Gamma_3(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} u\right\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-mp\beta} \right\},$$

$$\Gamma_4(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} v\right\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-mq\alpha} \right\}.$$

For fixed $l \in \{1, 2, \cdots, m-1\}$

$$\Gamma_{5l}(R) = \left\{ r \in (R, 2R) : \left\|u_{m-l-1}(r)\right\|_1 > KR^{-m\alpha-2(m-l-1)} \right\},$$

$$\Gamma_{6l}(R) = \left\{ r \in (R, 2R) : \left\|v_l(r)\right\|_1 > KR^{-m\beta-2l} \right\},$$

$$\Gamma_{7l}(R) = \left\{ r \in (R, 2R) : \left\|D_x u_{m-l-1}(r)\right\|_1 > KR^{-m\alpha-2(m-l-1)-1} \right\},$$

$$\Gamma_{8l}(R) = \left\{ r \in (R, 2R) : \left\|D_x v_l(r)\right\|_1 > KR^{-m\beta-2l-1} \right\}.$$

Since $\alpha + 2 = p\beta$, from Lemma 2.2 we deduce

$$cR^{N-mp\beta} \geq \int_R^{2R} \|v(r)\|_p^p r^{N-1} dr \geq \left|\Gamma_0^1(R)\right| KR^{-mp\beta} R^{N-1},$$

which implies

$$\left|\Gamma_0^1(R)\right| < \frac{1}{4m+8} R$$

for $K \gg 1$. Similarly, we get

$$\left|\Gamma_0^2(R)\right| < \frac{1}{4m+8} R$$

for $K \gg 1$.

To estimate $\Gamma_1(R)$, by (12) in Lemma 2.6,

$$\begin{aligned} CF(4R) &\geq \int_0^{2R} \left\|D_x^{2m}u\right\|_k^k r^{N-1}dr \\ &\geq |\Gamma_1(R)| KR^{-N}F(4R) R^{N-1} \\ &= |\Gamma_1(R)| KR^{-1}F(4R), \end{aligned}$$

From which it follows that for $K \gg 1$

$$|\Gamma_1(R)| < \frac{1}{4m+8}R.$$

Similarly we deduce from (13), (14) and (15) in Lemma 2.6 that

$$|\Gamma_2(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_3(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_4(R)| < \frac{1}{4m+8}R.$$

By (8) in Lemma 2.6,

$$\begin{aligned} CR^{N-m\alpha-2(m-l-1)} &\geq \int_0^{2R} \|u_{m-l-1}\|_1 r^{N-1}dr \\ &\geq |\Gamma_{5l}(R)| KR^{-m\alpha-2(m-l-1)}R^{N-1}, \end{aligned}$$

which gives

$$|\Gamma_{5l}(R)| < \frac{1}{4m+8}R$$

when $K \gg 1$ and similarly (9), (10) and (11) implies

$$|\Gamma_{6l}(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_{7l}(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_{8l}(R)| < \frac{1}{4m+8}R$$

when $K \gg 1$. In particular,

$$\Gamma(R) = (R, 2R) \setminus \left\{ \cup_{j=1}^2 \Gamma_0^j(R) \cup_{i=1}^4 \Gamma_i(R) \cup_{l=1}^{m-1} \cup_{j=5}^8 \Gamma_{jl}(R) \right\} \neq \emptyset.$$

Pick $\widetilde{R} \in \Gamma(R)$, by (33) together with the observation that $u_m = v^p$, $v_m = u^q$, we have

$$\begin{aligned} G_1\left(\widetilde{R}\right) \leq & C\widetilde{R}^N \sum_{l=0}^m \int_{S^{N-1}} |u_l|\,|v_{m-l}| \\ \leq & C\widetilde{R}^{N+2m} \sum_{l=0}^m \left\{ \left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon} + \widetilde{R}^{-2m+2l} \|u_l\|_1 \right)^{\nu_{1l}} \right. \\ & \cdot \left(\left\|D_x^{2m-2l}u_l\right\|_k + \widetilde{R}^{-2m+2l} \|u_l\|_1 \right)^{1-\nu_{1l}} \\ & \cdot \left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon} + \widetilde{R}^{-2l} \|v_{m-l}\|_1 \right)^{\nu_{2l}} \\ & \left. \cdot \left(\left\|D_x^{2l}v_{m-l}\right\|_d + \widetilde{R}^{-2l} \|v_{m-l}\|_1 \right)^{1-\nu_{2l}} \right\} \\ \leq & CR^{N+2m} \sum_{l=0}^m \left\{ R^{\frac{-mp\beta\nu_{1l}}{1+\varepsilon}} \left(R^{-N}F(4R)\right)^{\frac{1-\nu_{1l}}{k}} \left(R^{-\frac{2m+m\beta}{1+\varepsilon}} + R^{-2m-m\beta}\right)^{\nu_{2l}} \right. \\ & \left. \cdot \left(R^{-\frac{N}{d}}F(4R)^{\frac{1}{d}} + R^{-2m-m\beta}\right)^{1-\nu_{2l}} \right\} \\ \leq & CR^{-\widehat{a}}F^{\widehat{b}}(4R), \end{aligned}$$

with

$$\begin{aligned}
\widehat{a} &= \widehat{a}_\varepsilon = \min_l\{-N-2m+\frac{mp\beta}{1+\varepsilon}\nu_{1l}+\frac{mq\alpha}{1+\varepsilon}\nu_{2l}+\frac{N}{k}\left(1-\nu_{1l}\right)+\frac{N}{d}\left(1-\nu_{2l}\right)\},\\
\widehat{b} &= \max_l \frac{1}{k}\left(1-\nu_{1l}\right)+\frac{1}{d}\left(1-\nu_{2l}\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&G_2\left(\widetilde{R}\right)\\
\leq &CR^{N+2m}\sum_{l=1}^{m-1}\left\{\left(\left\|D_x^{2l+1}D_xu_{m-l-1}\right\|_{1+\varepsilon}+R^{-2l-1}\left\|D_xu_{m-l-1}\right\|_1\right.\right.\\
&\left.+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}}\\
&\cdot\left(\left\|D_x^{2l+1}D_xu_{m-l-1}\right\|_k+R^{-2l-1}\left\|D_xu_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}}\\
&\cdot\left(\left\|D_x^{2m-2l-1}D_xv_l\right\|_{1+\varepsilon}+R^{-2m+2l+1}\left\|D_xv_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{\tau_{2l}}\\
&\left.\cdot\left(\left\|D_x^{2m-2l-1}D_xv_l\right\|_d+R^{-2m+2l+1}\left\|D_xv_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{1-\tau_{2l}}\right\}\\
\leq &CR^{N+2m}\sum_{l=1}^{m-1}\left\{\left(\left\|D_x^{2m}u\right\|_{1+\varepsilon}+R^{-2l-1}\left\|D_xu_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}}\right.\\
&\cdot\left(\left\|D_x^{2m}u\right\|_k+R^{-2l-1}\left\|D_xu_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}}\\
&\cdot\left(\left\|D_x^{2m}v\right\|_{1+\varepsilon}+R^{-2m+2l+1}\left\|D_xv_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{\tau_{2l}}\\
&\left.\cdot\left(\left\|D_x^{2m}v\right\|_d+R^{-2m+2l+1}\left\|D_xv_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{1-\tau_{2l}}\right\}\\
\leq &CR^{N+2m}\sum_{l=1}^{m-1}\left\{\left(R^{-\frac{mp\beta}{1+\varepsilon}}+R^{-2l-1}R^{-m\alpha-2(m-l-1)-1}+R^{-2l-2}R^{-m\alpha-2(m-l-1)}\right)^{\tau_{1l}}\right.\\
&\cdot\left(R^{-N/k}F^{\frac{1}{k}}\left(4R\right)+R^{-2l-1}R^{-m\alpha-2(m-l-1)-1}+R^{-2l-2}R^{-m\alpha-2(m-l-1)}\right)^{1-\tau_{1l}}\\
&\cdot\left(R^{-\frac{mq\alpha}{1+\varepsilon}}+R^{-2l-1}R^{-m\beta-2(m-l-1)-1}+R^{-2l-2}R^{-m\beta-2(m-l-1)}\right)^{\tau_{2l}}\\
&\left.\cdot\left(R^{-N/d}F^{\frac{1}{d}}\left(4R\right)+R^{-2l-1}R^{-m\beta-2(m-l-1)-1}+R^{-2l-2}R^{-m\beta-2(m-l-1)}\right)^{1-\tau_{2l}}\right\}\\
\leq &CR^{-\overline{a}}F^{\overline{b}}\left(4R\right).
\end{aligned}$$

Here

$$\overline{a}=\overline{a}_\varepsilon=\min_l\{-N-2m+\frac{mp\beta}{1+\varepsilon}\tau_{1l}+\frac{mq\alpha}{1+\varepsilon}\tau_{2l}+\frac{N}{k}\left(1-\tau_{1l}\right)+\frac{N}{d}\left(1-\tau_{2l}\right)\}$$

and

$$\overline{b}=\max_l\frac{1}{k}\left(1-\tau_{1l}\right)+\frac{1}{d}\left(1-\tau_{2l}\right).$$

We claim that there exists a constant $M>0$ and a sequence $R_i\to\infty$ such that

$$F\left(4R_i\right)\leq MF\left(R_i\right).$$

Otherwise for any $M>0$, there exists R_M such that for $R\geq R_M$

$$F\left(4R\right)>MF\left(R\right).$$

Since (u, v) is bounded, we have $F(R) \le CR^N$, $R > 0$. Thus

$$M^i F(R_M) \le F(4^i R_M) \le CR_M^N (4^N)^i .$$

Contradiction for i large if $M > 4^N$.

Assume we have shown $a = a_\varepsilon = \min(\widehat{a_\varepsilon}\ ,\overline{a_\varepsilon}) > 0$, $b = b_\varepsilon = \max\left(\widehat{b},\ \overline{b}\right) < 1$,we have

$$\begin{aligned} F(R_i) &\le CR^{-a} F^b(4R_i) \\ &\le CM^b R^{-a} F^b(R_i), \end{aligned}$$

which gives

$$F(R_i) \le CR_i^{-\frac{a}{1-b}}.$$

Letting $i \to \infty$, we deduce that

$$\int_{\mathbb{R}^n} \left[u^{q+1} + v^{p+1}\right] dx = 0,$$

hence $u = v \equiv 0$, a contradiction.

Step 4. If $m\alpha > N - 2m - 1$, then $\overline{b}, \widehat{b} < 1$ and $\overline{a_\varepsilon}, \widehat{a_\varepsilon} > 0$ for $\varepsilon \ll 1$.

First we show $\widehat{a_\varepsilon} > 0$, $\widehat{b} < 1$. Since $\nu_{1l} = \left(\frac{1}{\delta_l} - \frac{1}{\gamma_l}\right)^{-1}\left(\frac{1}{\alpha_l} - \frac{1}{\gamma_l}\right)$, $\nu_{2l} = \left(\frac{1}{\psi_l} - \frac{1}{\omega_l}\right)^{-1}\left(\frac{1}{\alpha_l'} - \frac{1}{\omega_l}\right)$, to show $\widehat{b} < 1$, we need to show for all l,

$$\begin{aligned} & \frac{1}{k}(1 - \nu_{1l}) + \frac{1}{d}(1 - \nu_{2l}) \\ = \ & p\widehat{A}_{1l} + q\widehat{A}_{2l} \\ = \ & p\left(\frac{N - 2m + 2l - 1}{N - 1} - \frac{1}{\alpha_l}\right) + q\left(\frac{1}{\alpha_l} - \frac{2l}{N - 1}\right) < 1. \end{aligned} \tag{44}$$

Here

$$\begin{aligned} \widehat{A}_{1l} &= \frac{1}{p+1}(1 - \nu_{1l}) \\ &= \left(\frac{N - 2m + 2l - 1}{N - 1} - \frac{1}{\alpha_l}\right), \\ \widehat{A}_{2l} &= \frac{1}{q+1}(1 - \nu_{2l}) \\ &= \left(\frac{1}{\alpha_l} - \frac{2l}{N - 1}\right). \end{aligned}$$

(44) is equivalent to

$$p\frac{N - 2m + 2l - 1}{N - 1} - q\frac{2l}{N - 1} - 1 < \frac{p - q}{\alpha_l}. \tag{45}$$

Recall α_l is chosen to satisfy (26). Such $\alpha_l \in (1, \infty)$ satisfying (26) and (45) exists provided

$$\max\left(\frac{p}{p+1} - \frac{2m - 2l}{N - 1}, \frac{2l}{N - 1}\right) \le \min\left(1 - \frac{2m - 2l}{N - 1}, \frac{1}{q+1} + \frac{2l}{N - 1}\right) \tag{46}$$

and

$$p\frac{N - 2m + 2l - 1}{N - 1} - q\frac{2l}{N - 1} - 1 < (p - q)\left(1 - \frac{2m - 2l}{N - 1}\right), \tag{47}$$

$$p\frac{N-2m+2l-1}{N-1}-q\frac{2l}{N-1}-1<(p-q)\left(\frac{1}{q+1}+\frac{2l}{N-1}\right). \quad (48)$$

(46) follows from the assumption that $N\geq 2m+1$ and

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2m}{N}>1-\frac{2m}{N-1}.$$

(47) is equivalent to

$$q\frac{N-1-2m}{N-1}<1,$$

which follows from

$$q\leq\frac{p\,(q+1)}{p+1}=1+\frac{2}{\alpha}<1+\frac{2m}{N-2m-1}=\frac{N-1}{N-1-2m}.$$

And (48) can be rewritten as

$$\frac{N-2m-1}{N-1}p<\frac{p+1}{q+1},$$

which is equivalent to

$$m\alpha>N-2m-1.$$

Finally, for each l

$$\begin{aligned}\widehat{a_{0l}} &= -N-2m+mp\beta\nu_{1l}+mq\alpha\nu_{2l}+\frac{N}{k}\left(1-\nu_{1l}\right)+\frac{N}{d}\left(1-\nu_{2l}\right)\\ &= 2m-N+m\alpha+m\beta+\left(N-\frac{2m\,(p+1)\,(q+1)}{pq-1}\right)\left(p\widehat{A}_{1l}+q\widehat{A}_{2l}\right)\\ &= (2m-N+m\alpha+m\beta)\left(1-p\widehat{A}_{1l}-q\widehat{A}_{2l}\right)\\ &> 0.\end{aligned}$$

It then follows $\widehat{a_0}>0$, thus $\widehat{a_\varepsilon}>0$ for $\varepsilon\ll 1$.

Secondly we show $\overline{a_\varepsilon}>0$, $\overline{b}<1$. This can be shown in a similar way as $\widehat{a_\varepsilon},\widehat{b}$. We write all details for readers' convenience. Since $\tau_{1l}=\left(\frac{1}{\eta_l}-\frac{1}{\rho_l}\right)^{-1}\left(\frac{1}{z_l}-\frac{1}{\rho_l}\right)$, $\tau_{2l}=\left(\frac{1}{\kappa_l}-\frac{1}{\sigma_l}\right)^{-1}\left(\frac{1}{z_l'}-\frac{1}{\sigma_l}\right)$, to show $\overline{b}<1$, we need to show for all l,

$$\begin{aligned}&\frac{1}{k}\left(1-\tau_{1l}\right)+\frac{1}{d}\left(1-\tau_{2l}\right)\\ =\ & pA_{1l}+qA_{2l}\\ =\ & p\left(\frac{N-2l-2}{N-1}-\frac{1}{z_l}\right)+q\left(\frac{1}{z_l}-\frac{2m-2l-1}{N-1}\right)<1. \quad (49)\end{aligned}$$

Here we used

$$\begin{aligned}A_{1l} &= \frac{1}{p+1}\left(1-\tau_{1l}\right)\\ &= \left(\frac{N-2l-2}{N-1}-\frac{1}{z_l}\right),\\ A_{2l} &= \frac{1}{q+1}\left(1-\tau_{2l}\right)\\ &= \left(\frac{1}{z_l}-\frac{2m-2l-1}{N-1}\right).\end{aligned}$$

(49) is equivalent to

$$p\frac{N-2l-2}{N-1}-q\frac{2m-2l-1}{N-1}-1<\frac{p-q}{z_l}. \tag{50}$$

Recall z_l is chosen to satisfy (39) . Such $z_l \in (1,\infty)$ satisfying (39) and (50) exists provided

$$\max\left(\frac{p}{p+1}-\frac{2l+1}{N-1},\frac{2m-2l-1}{N-1}\right)\leq \min\left(1-\frac{2l+1}{N-1},\frac{1}{q+1}+\frac{2m-2l-1}{N-1}\right), \tag{51}$$

and

$$p\frac{N-2l-2}{N-1}-q\frac{2m-2l-1}{N-1}-1<(p-q)\left(1-\frac{2l+1}{N-1}\right), \tag{52}$$

$$p\frac{N-2l-2}{N-1}-q\frac{2m-2l-1}{N-1}-1<(p-q)\left(\frac{1}{q+1}+\frac{2m-2l-1}{N-1}\right). \tag{53}$$

(51) follows from

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2m}{N}>1-\frac{2m}{N-1}.$$

(52) is equivalent to

$$q\frac{N-1-2m}{N-1}<1,$$

which follows from

$$q\leq\frac{p\left(q+1\right)}{p+1}=1+\frac{2}{\alpha}<1+\frac{2m}{N-2m-1}=\frac{N-1}{N-1-2m}.$$

And lastly (53) can be rewritten as

$$\frac{N-2m-1}{N-1}p<\frac{p+1}{q+1},$$

which is equivalent to

$$m\alpha>N-2m-1.$$

Finally, for each l

$$\begin{aligned}\overline{a_{0l}}=&-2m-N+mp\beta\left(1-(p+1)A_{1l}\right)+mq\alpha\left(1-(q+1)A_{2l}\right)+NpA_{1l}+NqA_{2l}\\ =&2m-N+m\alpha+m\beta+\left(N-\frac{2m\left(p+1\right)\left(q+1\right)}{pq-1}\right)\left(pA_{1l}+qA_{2l}\right)\\ =&\left(2m-N+m\alpha+m\beta\right)\left(1-pA_{1l}-qA_{2l}\right)\\ >&0.\end{aligned}$$

It then follows $\overline{a_0}>0$, thus $\overline{a_\varepsilon}>0$ for $\varepsilon\ll 1$.

Remark 3. Note Lemma 2.2 implies when $pq>1$, $N\leq 2m$, (1) does not admit any positive solutions. In particular, this implies the following equation

$$\left(-\Delta\right)^m u=u^p$$

admits no positive solutions if $N\leq 2m$, $p>1$.

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