A Liouville-type theorem for higher order elliptic systems

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1. Introduction

In this paper, we consider positive solutions \((u > 0, \ v > 0)\) of the following higher order elliptic system

\[
\begin{align*}
(-\Delta)^m u &= v^p \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^m v &= u^q
\end{align*}
\]  \hspace{1cm} (1.1)

where \(p > 0, \ q > 0\) and \(N \geq 3\). We are mainly concerned with the question of nonexistence of such positive solutions.

When \(m = 1\), (1.1) becomes Lane–Emden system

\[
\begin{align*}
\Delta u + v^p &= 0 \\
\Delta v + u^q &= 0
\end{align*}
\]  \hspace{1cm} (1.2)

It has been conjectured that the curve \(\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}\) is the dividing curve for existence and nonexistence of positive solutions of (1.2). The conjecture was completely solved in the case of radial solutions [4,7,9]. Mitidieri [4] showed that there is no positive radial solutions to (1.2) below the curve \(\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}\) if \(p > 1, \ q > 1\); the condition \(p > 1, \ q > 1\) was later relaxed to \(p > 0, \ q > 0\) by Serrin and Zou [7,9]. Furthermore, it is proved by Serrin and Zou [9] that there are infinitely many positive radial solutions above the curve \(\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}\). Therefore \(\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}\) serves as the dividing curve for existence and nonexistence of positive radial solutions of (1.2).

The question for the general positive solution to (1.2), to the best of our knowledge, has not been completely solved yet. Partial answers have been given over the years. Souto [11] proved nonexistence of positive \(C^2\) solutions below the curve \(\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}\) when \(p, q > 0\). Felmer and de Figueredo [2] showed that when \(0 < p, q \leq \frac{N+2}{N-2}\) and \((p, q) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right)\),...
(1.2) has no positive $C^2$ solutions. Further evidence supporting the conjecture can be found in [5], where it is shown that there exists no positive supersolutions to (1.2) below the curve
\[
\left\{ p > 0, q > 0; \frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N-2} \max\left( \frac{1}{p+1}, \frac{1}{q+1} \right) \right\}.
\] (1.3)

We refer to (1.3) as S curve and the hyperbola in the conjecture $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ will be referred as Sobolev’s hyperbola throughout the paper. For $0 < p, q$, if $pq \leq 1$ or $pq > 1$ and $\max\left( \frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1} \right) \geq N - 2$, nonexistence of positive solutions was proved by Serrin and Zou in [8]. Direct calculation shows this is the same range of $(p, q)$ as region below and on S curve. Furthermore, Serrin and Zou [8] showed (1.2) admits no positive solutions satisfying algebraic growth at infinity below the Sobolev hyperbola when $N = 3$. For the special case $\min(p, q) = 1$, the conjecture was proved by C.-S. Lin [3]. Busca and Manásevich [1] proved that if $p, q > 0$, $pq > 1$,
\[
\frac{N-2}{2} \leq \min\left( \frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1} \right) \leq \max\left( \frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1} \right) < N - 2,
\]
and
\[
\left( \frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1} \right) \neq \left( \frac{N-2}{2}, \frac{N-2}{2} \right),
\]
there is no positive classical solutions to (1.2). Most recently, the conjecture was fully solved in the case $N = 3$ by Poláčik, Quittner and Souplet [6] and by Souplet [10] when $N = 4$. Souplet also proved the conjecture when $N \geq 5$ under the additional assumption that $\max\left( \frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1} \right) > N - 3$.

Comparing to the Lane–Emden system, less is known about the higher order system (1.1). In the single equation case, Mitidieri [4] proved that for $1 < q < \frac{N+m}{N-4m}$, $N > 4m$, the problem
\[
\begin{cases}
\Delta^{2m} u = u^q, \\
(-\Delta)^s u \geq 0, \quad s = 1, 2, \ldots, 2m - 1
\end{cases}
\]
has no nontrivial positive radial solution of class $C^{4m}(\mathbb{R}^N)$. In this paper, we prove the following generalization of the Liouville-type theorem to higher order elliptic system. Our first Liouville-type theorem deals with radially symmetric positive solutions of (1.1).

**Theorem 1.1.** If $N \geq 3$, $N > 2m$, $p \geq 1$, $q \geq 1$, $(p, q) \neq (1, 1)$ and $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2m}{N}$, the problem (1.1) has no nontrivial positive radial solutions of class $C^{2m}(\mathbb{R}^N)$.

Our second theorem handles Liouville properties of general solutions of (1.1).

**Theorem 1.2.** $N \geq 3$, $N > 2m$, if $p \geq 1$, $q \geq 1$, $(p, q) \neq (1, 1)$ and
\[
\max\left( \frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1} \right) \geq N - 2m,
\]
the problem (1.1) has no nontrivial positive solutions of class $C^{2m}(\mathbb{R}^N)$.

The assumption $p \geq 1$, $q \geq 1$, $(p, q) \neq (1, 1)$ in the previous two theorems is only needed to show that any positive solution $(u, v)$ of (1.1) satisfies
\[
(-\Delta)^i u > 0, \quad (-\Delta)^i v > 0, \quad i = 1, 2, \ldots, m - 1.
\]

We shall prove the following version of the radial and general case and Theorem 1.1 and Theorem 1.2 are obtained as corollaries of the following theorems respectively.

**Theorem 1.3.** If $N \geq 3$, $N > 2m$, and $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2m}{N}$, then the problem
\[
\begin{cases}
(-\Delta)^m u = v^p \\
(-\Delta)^m v = u^q \\
(-\Delta)^i u > 0, \quad (-\Delta)^i v > 0, \quad i = 1, 2, \ldots, m - 1
\end{cases}
\] in $\mathbb{R}^N$,

has no nontrivial positive radial solutions of class $C^{2m}(\mathbb{R}^N)$.
Theorem 1.4. Let $N \geq 3$, $N > 2m$, then the problem
\begin{equation}
\begin{aligned}
(-\Delta)^m u &= v^p \\
(-\Delta)^m v &= u^q
\end{aligned}
\end{equation}
in $\mathbb{R}^N$, has no nontrivial positive solutions of class $C^m(\mathbb{R}^N)$ if $pq \leq 1$ or if $pq > 1$ and $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) \geq N - 2m$.

The paper is organized as follows. Section 2 presents proof of radial case (Theorem 1.1 and Theorem 1.3), Section 3 is devoted to the proof of general case (Theorem 1.2 and Theorem 1.4). The proof of the radial case uses Rellich’s Identity and the proof of the general solution case relies on growth estimates of the spherical average of the solution.

2. Radial solutions

First we recall the following function defined in [4]

\[ R_n(u, v) = \int_{\Omega} \Delta^n u(x, \nabla v) + \Delta^n v(x, \nabla u) \, dx \]

where $u, v \in C^2(\mathbb{R})$, $n \geq 1$. If $n = 1$, we have

\[ R_1(u, v) = \int_{\partial \Omega} \frac{\partial u}{\partial n}(x, \nabla v) + \frac{\partial v}{\partial n}(x, \nabla u) - (\nabla u, \nabla v)(x, n) \, ds + (N - 2) \int_{\Omega} (\nabla u, \nabla v) \, dx. \]

If $n = 2$,

\[ R_2(u, v) = R_1(A u, v) + R_1(u, A v) - B(u, v) \tag{2.1} \]

where

\[ B(u, v) = \int_{\partial \Omega} \Delta u \Delta v(x, n) \, ds - N \int_{\Omega} \Delta u \Delta v \, dx. \tag{2.2} \]

We quote the following lemma from [4]

Lemma 2.1. (See [4, Lemma 2.2].) If $u, v \in C^2(\mathbb{R})$, then for $1 \leq s \leq n - 2$

\[ R_n(u, v) = \sum_{k=0}^{s} R_{n-s} (\Delta^k u, \Delta^s v) - \sum_{k=0}^{s-1} R_{n-(s+1)} (\Delta^{k+1} u, \Delta^{s-k} v). \tag{2.3} \]

Remark 2.2. An immediate consequence of Lemma 2.1 is the following implicit form of Rellich’s identity. If $u, v \in C^2(\mathbb{R})$, then

\[ R_n(u, v) = \sum_{k=0}^{n-1} R_{1} (\Delta^k u, \Delta^{n-k} v) - \sum_{k=0}^{n-2} B(\Delta^k u, \Delta^{n-2-k} v). \tag{2.4} \]

Proof. Choose $s = n - 2$ in (2.3), taking into account of (2.1) and (2.2), (2.4) follows. $\square$

Lemma 2.3. (See [4, Lemma 3.1].) If $\psi \in C^2(\mathbb{R}^N)$ ($N \geq 3$) is positive, radial and superharmonic (i.e. $\Delta \psi \leq 0$ in $\mathbb{R}^N$), then

\[ r \psi'(r) + (N - 2) \psi(r) \geq 0 \quad \text{for } r > 0. \]

For $w \in C(\mathbb{R}^N)$, denote the spherical average of $w$ by

\[ \bar{w}(r) = \frac{1}{\omega_n} \int_{S^{N-1}} w(r, \theta), \quad r > 0, \]

where $(r, \theta)$ are spherical coordinates and $\omega_n = |S^{N-1}|$ is the area of the unit sphere in $\mathbb{R}^N$. We quote the following lemma from [8].
Lemma 2.4. (See [8, Lemma 2.3].) Suppose that \( w \geq 0 \) is nontrivial and satisfies
\[
\Delta w \leq 0, \quad x \in \mathbb{R}^N.
\]
Then for \( a \in (0, 1] \), \((\overline{w^a})' \leq 0\). In particular, \( \overline{w^a} \) is a nonincreasing function of \( r \).

Lemma 2.5. Let \((u, v)\) be a pair of positive solutions of (1.1), if \( p \geq 1, q \geq 1, (p, q) \neq (1, 1)\), then we have
\[
(-\Delta)^i u > 0, \quad (-\Delta)^i v > 0, \quad i = 1, 2, \ldots, m - 1.
\]

Proof. We follow an idea in [12]. Let \( u_i = (-\Delta)^i u, v_i = (-\Delta)^i v \), \( i = 0, 1, \ldots, m - 1 \) with \( u_0 = u, v_0 = v \). We first prove \( v_{m-1} > 0 \). Suppose not, there exists \( x_0 \in \mathbb{R}^n \) such that
\[
v_{m-1}(x_0) < 0.
\]
Without loss of generality, we assume \( x_0 = 0 \). Then we have, since \( p \geq 1, q \geq 1, \)
\[
\Delta u + \overline{u_1} = 0, \\
\Delta \overline{u_1} + \overline{u_2} = 0, \\
\vdots \\
\Delta \overline{u_{m-1}} + \overline{u^p} \leq 0.
\]
and
\[
\Delta v + \overline{v_1} = 0, \\
\Delta \overline{v_1} + \overline{v_2} = 0, \\
\vdots \\
\Delta \overline{v_{m-1}} + \overline{v^q} \leq 0.
\]
Since \( v_{m-1}(0) < 0 \) and \( \overline{v_{m-1}}' < 0 \), we have
\[
v_{m-1}(r) < 0 \quad \text{for all } r > \overline{r_1} = 0.
\]
It then follows easily that
\[
\overline{v_{m-2}}' > -\frac{v_{m-1}(0)}{n}r.
\]
hence
\[
\overline{v_{m-2}}(r) \geq c_2 r^2, \quad \text{for } r > \overline{r_2} > \overline{r_1}.
\]
The same arguments show that
\[
\overline{v_{m-3}}(r) \leq -c_3 r^4, \quad \text{for } r > \overline{r_3} > \overline{r_2}.
\]
and
\[
(-1)^i \overline{v_{m-i}}(r) \geq c_i r^{2(i-1)}, \quad \text{for } r > \overline{r_i}, \quad i = 1, \ldots, m.
\]
Hence if \( m \) is odd, we have a contradiction with the fact that \( v > 0 \).
So \( m \) must be even and we have
\[
\overline{v} \geq c_0 r^{\sigma_0}, \quad \sigma_0 = 2(m - 1)
\]
and
\[
(-1)^i \overline{v_{m-i}} > 0
\]
for \( r > \overline{r_0} > 0 \).

Setting \( A = 2^{q+1}(n + 2m + 2mq + 2pq(m - 1))^{1+q} \) and suppose that
\[
\overline{v}(r) \geq c_0^p \frac{r^{\sigma_k}}{\overline{A^{b_k}}}, \quad \text{for } r \geq r_k.
\]
Then we have
\[
r^{n-1} \overline{u_{m-1}}(r) \leq r^{n-1} \overline{u_{m-1}}(r_k) - \int_{r_k}^r s^{n-1} \overline{v^p}(s) \, ds.
\]
therefore
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{1+p\sigma_k} - r^{1+p\sigma_k}}{A^{p\sigma_k}(p\sigma_k + n)} c_0^0 \rho_{p+1}, \]

hence
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{1+p\sigma_k}}{2A^{p\sigma_k}(p\sigma_k + n)} c_0^0 \rho_{p+1} \]

for \( r \geq 2^{\frac{1}{p\sigma_k + 1}} r_k \). Similarly
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{2+p\sigma_k}}{4A^{p\sigma_k}(p\sigma_k + n)(n+2m+p\sigma_k + 2)} c_0^0 \rho_{p+1} \]

for \( r \geq 2^{\frac{1}{2p\sigma_k + 1}} r_k \).

Hence
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{2+p\sigma_k}}{4A^{p\sigma_k}(p\sigma_k + n)^2} c_0^0 \rho_{p+1} \]

for \( r \geq 2^{\frac{2}{p\sigma_k}} r_k \).

By induction, we have
\[ (-1)^i \bar{u}_{m-i}(r) \geq \frac{c_0^0 \rho_{p+1}}{2^{2i} + p\sigma_k (n+2m+p\sigma_k)^{2i} A^{p\sigma_k} 4^i} \]

for \( r \geq 2^{\frac{2i}{p\sigma_k}} r_k \).

In particular,
\[ \bar{u}(r) \geq \frac{c_0^0 \rho_{p+1}}{(n+2m+p\sigma_k)^{2m} A^{p\sigma_k} 4^m} \]

for \( r \geq 2^{\frac{2m}{p\sigma_k}} r_k = s_k \).

Since
\[ r^{n-1} \bar{u}_{m-1}(r) \leq s_k^{n-1} \bar{u}_{m-1}(s_k) - \int_{s_k}^r s^{n-1} \bar{u}(s) \, ds, \]

which implies
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{1+2mq+p\sigma_k} - s_k^{1+2mq+p\sigma_k}}{(n+2m+p\sigma_k)^{2mq} (n+2m+p\sigma_k)^{2mq} A^{p\sigma_k} 4^{mq} c_0^0 \rho_{p+1}^{(2mq)}}, \]

hence
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{1+2mq+p\sigma_k}}{2^{2mq} A^{p\sigma_k} (n+2m+p\sigma_k)^{2mq} (n+2m+p\sigma_k)^{2mq} c_0^0 \rho_{p+1}^{(2mq)}}, \]

for \( r \geq 2^{\frac{1}{2mq+p\sigma_k}} r_k \). Similarly
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{2+2mq+p\sigma_k}}{4^{mq+1} A^{p\sigma_k} (n+2m+p\sigma_k)^{2mq} (n+2m+p\sigma_k)^{2mq} c_0^0 \rho_{p+1}^{(2mq)}}, \]

for \( r \geq 2^{\frac{1}{2mq+p\sigma_k}} r_k \).

Hence
\[ \bar{u}_{m-1}(r) \leq - \frac{r^{2+2mq+p\sigma_k}}{4^{mq+1} A^{p\sigma_k} (n+2m+p\sigma_k)^{2mq} (pq\sigma_k + n + 2mq)^2 c_0^0 \rho_{p+1}^{(2mq)}}, \]

for \( r \geq 2^{\frac{2}{2mq+p\sigma_k}} r_k \).

By induction, we have
\[ (-1)^i \bar{u}_{m-i}(r) \geq \frac{c_0^0 \rho_{p+1}}{(n+2m+2mq+p\sigma_k)^{2i} (n+2m+p\sigma_k)^{2mq} A^{p\sigma_k} 4^i +mq} \]

\[ (n+2m+2mq+p\sigma_k)^{2i} (n+2m+p\sigma_k)^{2mq} A^{p\sigma_k} 4^i +mq} \]
for \( r \geq 2^{\frac{2m}{n+2m+2mq+pq\sigma_k}} s_k \), Hence
\[
\bar{v}(r) \geq \frac{c_0^{(pq)^{k+1}} r^{2m+2mq+pq\sigma_k}}{(n+2m+2mq+pq\sigma_k)^{2m}(n+2m+pq\sigma_k)^{2mq} A_p^p B_q^q A_m^{m+mq}}
\] (2.5)

for \( r \geq 2^{\frac{2m}{n+2m+2mq+pq\sigma_k}} s_k \). Set
\[
\sigma_0 = 2(m-1), \quad r_0 = \bar{r}_0,
\]
\[
\sigma_{k+1} = 2m + 2mq + pq\sigma_k,
\]
\[
r_{k+1} = 2^{\frac{2m}{n+2m+2mq+pq\sigma_k}} r_k.
\] (2.6)

By our choice of \( A \), one can show via induction that
\[
(n + 2m + 2mq + pq\sigma_k)^{2m} (n + 2m + pq\sigma_k)^{2mq} A_m^{m+mq} \leq A^{2m(k+1)}.
\]

We can then set
\[
b_0 = 0, \quad b_{k+1} = pq b_k + 2m(k+1),
\] (2.7)

and rewrite (2.5) as
\[
\bar{v}(r) \geq \frac{c_0^{(pq)^{k+1}} r^\sigma_{k+1}}{A_{b_{k+1}}^r}, \quad r \geq r_{k+1}.
\]

Notice that
\[
r_{k+1} \leq c r_0,
\]

where
\[
c = 2^{\sum_{k=0}^{\infty} \frac{2m}{n+2m+pq\sigma_k} + \frac{2m}{n+2m+2mq+pq\sigma_k}}.
\]

From iteration formula (2.6), (2.7), we have
\[
\sigma_k = 2(m-1)(pq)^k + 2m(q+1)(pq)^k - 1, \quad \frac{pq k - 1}{pq - 1},
\]
\[
b_k = 2m \frac{(pq)^{k+1} - (k+1) pq + k}{(pq - 1)^2}.
\]

Take \( M > 1 \) large enough so that \( M A_m^{\frac{2m}{n+2m+pq\sigma_k}} \geq 2cr_0 \) if \( c_0 \geq 1 \) and \( M A_m^{\frac{2m}{n+2m+2mq+pq\sigma_k}} c_0^{-1} \geq 2r_0 \) if \( c_0 < 1 \), and take \( r_1 = MA_m^{\frac{2m}{n+2m+2mq+pq\sigma_k}} \) or \( M A_m^{\frac{2m}{n+2m+2mq+pq\sigma_k}} c_0^{-1} \) depending on whether \( c_0 \) is greater or less than 1, then we have
\[
\bar{v}(r_1) \geq \left[ A_m^{\frac{2m}{n+2m+2mq+pq\sigma_k}} \right]^{(6m-4+4mq)(pq^{k+1} - 4(m+mq) + 2m(k+1)pq - 2mk)} \to \infty \quad \text{as } k \to \infty,
\]
contradiction to the fact that \( r_1 \) is independent of \( k \).

Hence \( v_{m-1} > 0, u_{m-1} > 0 \) can be proved similarly. Next we claim \( v_{m-i} > 0, u_{m-i} > 0 \) for \( i = 2, \ldots, m - 1 \). Proof is exactly the same except that we need to take extra care when \( m \) is odd. We omit the details. \( \square \)

**Lemma 2.6.** If \( pq > 1, p > 1, q > 1, (u, v) \in C^{2m}(\mathbb{R}^N), N > 3, \) is a pair of positive radial solution of (1.1), then
\[
u(r) \leq Cr^{-\frac{2m(p+1)}{n+2m+2mq+pq\sigma_k}} \quad \text{as } k \to \infty, \quad r > 0,
\]

and for \( k = 0, 1, \ldots, m - 1, r > 0 \)
\[
|r^N \Delta^{k+1} u \Delta^{m-1-k} v| \leq Cr^{-2m-\frac{2m(p+q+2)}{pq-1}} \quad \text{as } k \to \infty,
\]
\[
|r^{N-1} (\Delta^k u)' \Delta^{m-1-k} v| \leq Cr^{-2m-\frac{2m(p+q+2)}{pq-1}} \quad \text{as } k \to \infty,
\]
\[
|r^{N-1} \Delta^k u (\Delta^{m-1-k} v)'| \leq Cr^{-2m-\frac{2m(p+q+2)}{pq-1}} \quad \text{as } k \to \infty,
\]
\[
|r^N (\Delta^k u)' (\Delta^{m-1-k} v)'| \leq Cr^{-2m-\frac{2m(p+q+2)}{pq-1}} \quad \text{as } k \to \infty.
\] (2.8)
Proof. Let \( u_k = (-\Delta)^k u, \) \( v_k = (-\Delta)^k v. \) By Lemma 2.5, \( u_k > 0, v_k > 0 \) and \(-\Delta u_k > 0, -\Delta v_k > 0, \) Lemma 2.4 implies \( u_k' < 0, v_k' < 0. \) And

\[
-(u_k'r^{N-1})' = u_{k+1}r^{N-1}, \quad -(v_k'r^{N-1})' = v_{k+1}r^{N-1}, \quad r > 0.
\]

(Integrating (2.9), and using the fact that \( u_k' < 0, v_k' < 0, \) we obtain

\[
-u_k'r^{N-1} = \frac{r^N}{N}u_{k+1} - \frac{1}{N} \int_0^r s^N u_{k+1}' ds \geq \frac{r^N}{N} u_{k+1},
\]

\[
-v_k'r^{N-1} = \frac{r^N}{N}v_{k+1} - \frac{1}{N} \int_0^r s^N v_{k+1}' ds \geq \frac{r^N}{N} v_{k+1}.
\]

An application of Lemma 2.3 to \( u_k, v_k \) yields

\[
r u_k' + (N-2) u_k \geq 0, \quad r v_k' + (N-2) v_k \geq 0.
\]

(2.10) and (2.13) that

\[
r^2 u_{k+1} \leq N(N-2) u_k, \quad k = 0, 1, \ldots, m - 1,
\]

and from (2.11), (2.13) that

\[
r^2 v_{k+1} \leq N(N-2) v_k, \quad k = 0, 1, \ldots, m - 1.
\]

(2.14) and (2.15) imply

\[
r^2 v^p = r^{2m} u_m \leq N^m (N-2)^m u,
\]

\[
r^2 u^q = r^{2m} v_m \leq N^m (N-2)^m v,
\]

from which we can easily solve if \( pq > 1 \) that

\[
u \leq C r^{-2m(p+1)} u^{-2m(p+1)}, \quad \nu \leq C r^{-2m(q+1)} v^{-2m(q+1)},
\]

and for \( k = 1, 2, \ldots, m - 1 \)

\[
u_k \leq C r^{-2k} C r^{-2k(p+1)} v^{-2k(p+1)}, \quad \nu_k \leq C r^{-2k} v^{-2k(q+1)} v^{-2k(q+1)}.
\]

(2.8) follows. \( \square \)

Proof of Theorem 1.1. We prove by contradiction. Let \( (u, v) \) be a pair of nontrivial positive radial solution of (1.1), we have

\[
-(u_{m-1}'r^{N-1})' = v^{p+1} v^N, \quad -(v_{m-1}'r^{N-1})' = u^{q+1} u^N.
\]

Multiply (2.16) by \( v \) and (2.17) by \( u \) and integrating by parts on \((0, r)\) we obtain

\[
-u_{m-1}'(r)v(r)r^{N-1} + \int_0^r u_{m-1}'(s)v'(s)s^{N-1} ds = \int_0^r v^{p+1}(s)s^{N-1} ds,
\]

\[
-v_{m-1}'(r)u(r)r^{N-1} + \int_0^r v_{m-1}'(s)u'(s)s^{N-1} ds = \int_0^r u^{q+1}(s)s^{N-1} ds.
\]

Using the fact that

\[
\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N} \quad \Leftrightarrow \quad \frac{N-2m}{2m} < \frac{p+q+2}{pq-1}
\]

and (2.8), we conclude

\[
u_{m-1}'(r)v(r)r^{N-1} \to 0, \quad v_{m-1}'(r)u(r)r^{N-1} \to 0.
\]
therefore
\[
\int_0^\infty u_{m-1}'(s)v(s)s^{N-1} ds = \int_0^\infty v^{p+1}(s)s^{N-1} ds,
\]
\[
\int_0^\infty v_{m-1}'(s)u(s)s^{N-1} ds = \int_0^\infty u^{q+1}(s)s^{N-1} ds.
\] (2.21)

On the other hand
\[
\frac{N}{p+1} \int_0^r v^{p+1}(s)s^{N-1} ds + \frac{N}{q+1} \int_0^r u^{q+1}(s)s^{N-1} ds
\]
\[
= \frac{1}{p+1} v^{p+1}(r)r^N + \frac{1}{q+1} u^{q+1}(r)r^N - \int_0^r v^{p}(s)v'(s)s^N ds - \int_0^r u^{q}(s)u'(s)s^N ds
\]
\[
= \frac{1}{p+1} v^{p+1}(r)r^N + \frac{1}{q+1} u^{q+1}(r)r^N - \frac{(-1)^m}{\omega_N} R_m(u, v).
\] (2.22)

Here we used
\[
(-1)^m R_m(u, v) = \int_{\bar{B}_r} (-\Delta)^m u(x, \nabla v) + (-\Delta)^m v(x, \nabla u) \, dx
\]
\[
= \int_{\bar{B}_r} v^{p}(x, \nabla v) + u^{q}(x, \nabla u) \, dx
\]
\[
= \omega_N \int_0^r v^{p}(s)v'(s)s^N + u^{q}(s)u'(s)s^N ds.
\]

By (2.4), we have
\[
R_m(u, v) = \sum_{k=0}^{m-1} R_1(\Delta^k u, \Delta^{m-1-k} v) - \sum_{k=0}^{m-2} B(\Delta^k u, \Delta^{m-2-k} v)
\]
\[
= \sum_{k=0}^{m-1} \int_{\partial \bar{B}_r} \frac{\partial \Delta^k u}{\partial n}(x, \nabla \Delta^{m-1-k} v) + \frac{\partial \Delta^{m-1-k} v}{\partial n}(x, \nabla \Delta^k u) \, dx
\]
\[
- \sum_{k=0}^{m-1} \int_{\partial \bar{B}_r} (\nabla \Delta^k u, \nabla \Delta^{m-1-k} v)(x, n) \, ds + (N-2) \sum_{k=0}^{m-1} \int_{\partial \bar{B}_r} (\nabla \Delta^k u, \nabla \Delta^{m-1-k} v) \, dx
\]
\[
- \sum_{k=0}^{m-2} \int_{\partial \bar{B}_r} (\Delta^{k+1} u, \Delta^{m-1-k} v)(x, n) \, ds + N \sum_{k=0}^{m-2} \int_{\partial \bar{B}_r} (\Delta^{k+1} u, \Delta^{m-1-k} v) \, dx.
\] (2.23)

Since
\[
\int_{\bar{B}_r} (\nabla \Delta^k u, \nabla \Delta^{m-1-k} v) \, dx = \int_{\bar{B}_r} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v \, ds - \int_{\bar{B}_r} (\Delta^{k+1} u, \Delta^{m-1-k} v) \, dx,
\]
we can rewrite (2.23) as
\[
R_m(u, v) = \sum_{k=0}^{m-1} \int_{\partial \bar{B}_r} \frac{\partial \Delta^k u}{\partial n}(x, \nabla \Delta^{m-1-k} v) + \frac{\partial \Delta^{m-1-k} v}{\partial n}(x, \nabla \Delta^k u) \, ds
\]
\[
- \sum_{k=0}^{m-2} \int_{\partial \bar{B}_r} (\Delta^{k+1} u, \Delta^{m-1-k} v)(x, n) ds + (N-2) \sum_{k=0}^{m-1} \int_{\partial \bar{B}_r} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v \, ds
\]
Theorem 1.1 is to show $u_k$ follows from the same argument above.

We first quote the following lemmas from [8].

Proof of Theorem 1.3. Notice the only place where assumption $p \geq 1, q \geq 1, (p, q) \neq (1, 1)$ is needed in the proof of Theorem 1.1 is to show $u_k = (-\Delta)^k u > 0, v_k = (-\Delta)^k v > 0$ for any given pair of positive solution $(u, v)$. Theorem 1.3 follows from the same argument above.

3. General solutions

When $pq > 1$, we introduce the following notation

$$\alpha = \frac{2(p + 1)}{pq - 1}, \quad \beta = \frac{2(q + 1)}{pq - 1}$$

and assume $\alpha \geq \beta$ throughout the rest of the section. The assumption

$$\frac{1}{p + 1} + \frac{1}{q + 1} > \frac{N - 2m}{m}$$

can be rewritten as

$$ma + mb > N - 2m.$$
Lemma 3.2. Let $u$ be a positive solution

$$\Delta u + v^p = 0,$$

where $v$ is a positive function, then

$$U'' + \frac{N-1}{r}U' + V^p \leq 0 \quad \text{for } 0 < p_1 \leq p$$

where

$$U_1(r) = \left(\frac{U}{r}\right)^{\frac{2}{p_1}}, \quad V_1(r) = \left(\frac{V}{r}\right)^{\frac{1}{p_1}}.$$ 

Proof. See proof of Lemma 2.6 in [8].

We can now prove the following growth estimate on $\bar{u}, \bar{v}$.

Lemma 3.3. If $pq = 1$, there is no nontrivial positive solution of (1.4). If $(u, v)$ is a positive solution of (1.4) and $pq > 1$, there exists a positive constant $M = M(p, q, n)$ such that

$$\bar{u}(r) \leq M r^{-\frac{2(p+1)}{pq-1}}, \quad \bar{v}(r) \leq M r^{-\frac{2(p+1)}{pq-1}} \text{ for } r > 0,$$

and for $k = 1, \ldots, m-1$, $u_k = (-\Delta)^k u$, $v_k = (-\Delta)^k v$

$$\bar{u}_k(r) \leq M r^{-\frac{2(p+1)}{pq-1}-2k}, \quad \bar{v}_k(r) \leq M r^{-\frac{2(p+1)}{pq-1}-2k} \text{ for } r > 0.$$ 

Proof. We first consider the case $p, q \geq 1$. Taking the spherical average of (1.4) and using Jensen’s inequality, we have

$$\bar{u}_{m-1}'' + \frac{N-1}{r} \bar{u}_{m-1}' + \bar{v}^p \leq 0,$$

$$\bar{v}_{m-1}'' + \frac{N-1}{r} \bar{v}_{m-1}' + \bar{u}^q \leq 0,$$

and

$$\bar{u}_{k-1}'' + \frac{N-1}{r} \bar{u}_{k-1}' + \bar{u}_k = 0,$$

$$\bar{v}_{k-1}'' + \frac{N-1}{r} \bar{v}_{k-1}' + \bar{v}_k = 0$$

for $k = 1, 2, \ldots, m-1$. By assumption, $\bar{u}_k, \bar{v}_k$ are nonincreasing for $k = 0, 1, \ldots, m-1$, therefore we get from Lemma 3.1

$$\bar{u}_{m-1} \geq cr^2 \bar{v}^p, \quad \bar{v}_{m-1} \geq cr^2 \bar{u}^q,$$

$$\bar{u}_{k-1} \geq cr^2 \bar{u}_k, \quad \bar{v}_{k-1} \geq cr^2 \bar{v}_k.$$ 

(3.3)

Solving (3.3), we obtain (3.1) and (3.2). In particular, we immediately get a contradiction if $pq = 1$.

Next we assume $pq \geq 1$ and $p < 1$. Taking spherical average of (1.1), we have

$$\bar{v}_{m-1}'' + \frac{N-1}{r} \bar{v}_{m-1}' + \bar{u}^q = 0.$$

From Lemma 3.1, we obtain

$$\bar{u}_{m-1} \geq cr^2 \bar{v}^p, \quad \bar{v}_{m-1} \geq cr^2 \bar{u}^q,$$

$$\bar{u}_{k-1} \geq cr^2 \bar{u}_k, \quad \bar{v}_{k-1} \geq cr^2 \bar{v}_k.$$ 

(3.4)

Solving (3.4), taking into account of Lemma 3.2 and that $(u^p)^q \leq (\bar{u}^q)^{\frac{q}{p}}$ since $q \geq \frac{1}{p}$. We get a contradiction when $pq = 1$ and if $pq > 1$ it follows that

$$\bar{u}^q \leq cr^{-m_0 q}, \quad \bar{v} \leq cr^{-m\beta}.$$

We can now follow the same argument in p. 644 of [8] to conclude (3.1), (3.2) then follows from (3.1) and (3.4).
Lemma 3.4. Suppose that \( pq > 1 \) and \((u, v)\) is a positive solution of (1.4). Then
\[
\int_{B_R} u^q \leq c R^{N - 2m - \frac{2(q+1)}{pq-1}}, \quad \int_{B_R} v^p \leq c R^{N - 2m - \frac{2(p+1)}{pq-1}},
\]
where \( c = c(p, q, n) \).

Proof.
\[
\int_{B_R} u^q \leq c R^{N - 2} \int_{r^{1-N}}^{2R} \int_{B_r} u^q = c R^{n-2} \int_{r^{1-N}}^{2R} \int_{B_r} v^p = c R^{N - 2} \int_{r^{1-N}}^{2R} \int_{B_r} \partial v_{m-1} \partial n
\]
\[
= -c R^{N - 2} \int_{r^{1-N}}^{2R} \int_{B_r} \partial v_{m-1} \partial n \leq c R^{N - 2} \int_{B_r} v_{m-1} \int_{B_r} u^q = c R^{N - 2} \int_{B_r} v_{m-1} \int_{B_r} u^q
\]
\[
\leq MR^{N - 2 - \frac{2(q+1)}{pq-1}} \int_{B_r} v_{m-1} = c R^{N - 2 - \frac{2(q+1)}{pq-1}}.
\]

the second inequality can be proved similarly. \( \square \)

Lemma 3.5. Suppose \( pq \leq 1 \). Then (1.4) has no nontrivial and non-negative solutions.

Proof. The case \( pq = 1 \) is proved in Lemma 3.3. We assume \( pq < 1 \). Choose \( s \) and \( \theta \) such that
\[
p < s < \frac{1}{q}, \quad 0 < \theta < \min \left( 1, \frac{1}{s} \right),
\]
and put \( \gamma = \theta s \). It follows that
\[
\theta p < \gamma < 1, \quad \gamma q < \theta < 1
\]
and
\[
a = \frac{1 - \gamma}{1 - \gamma q} \in (0, 1), \quad b = \frac{1 - \gamma}{1 - \theta p} \in (0, 1).
\]

By Hölder’s inequality,
\[
\bar{u}^\theta \leq (\bar{u}^{\gamma q})^{\theta} \bar{u}^{-\alpha} \leq c(\bar{u}^{\gamma q})^\theta,
\]
since \( \bar{u} \) is decreasing thus bounded. Similarly,
\[
\bar{v}^\gamma \leq c(\bar{v}^{\theta p})^b.
\]

On the other hand, apply Lemma 3.2 with \( p_1 = \theta p, q_1 = \gamma q \), we conclude from Lemma 3.1 that
\[
U_{m-1} = (\bar{u}^{\gamma q})^{\frac{1}{\theta}} \geq c r^2 (\bar{u}^{\gamma q})^{\frac{1}{\theta}}, \quad V_{m-1} = (\bar{v}^{\theta p})^{\frac{1}{\gamma}} \geq c r^2 (\bar{v}^{\theta p})^{\frac{1}{\gamma}}
\]
and similarly
\[
U_k = (\bar{u}^{\gamma q})^{\frac{1}{\theta}} \geq c r^2 (\bar{u}^{\gamma q})^{\frac{1}{\theta}}, \quad V_k = (\bar{v}^{\theta p})^{\frac{1}{\gamma}} \geq c r^2 (\bar{v}^{\theta p})^{\frac{1}{\gamma}}, \quad k = 0, 1, \ldots, m - 1.
\]

A short calculation then yields
\[
\bar{u}^\theta \leq c r^{-\frac{2n(m+1)(s+b)}{1-ab}}.
\]

Since \( \Delta \bar{u}^\theta \leq 0 \), it follows
\[
N - 2 \geq \frac{2n(m+1)(s+b)}{1-ab}.
\]
For \( s \) fixed, it is easy to see that
\[
\lim_{\theta \to 0} a = \lim_{\theta \to 0} b = 1.\]
Therefore
\[
\lim_{\theta \to 0} \frac{2a(m+1)(s+b)}{1-ab} = \infty,
\]
contradiction. Hence no solutions exist. \( \Box \)

**Proof of Theorem 1.4.** Case \( pq \leq 1 \) follows from Lemma 3.5. Assume \( pq > 1 \), suppose by contradiction that (1.1) admits a non-negative and nontrivial solution \((u, v)\). Then \( u > 0, v > 0 \) by maximum principle. If \( \alpha > \frac{N-2m}{m} \), it follows from (3.5) that
\[
\int_{B_R} v^p \leq cR^{N-2m-\frac{2m(p+1)}{pq-1}} \to 0 \quad \text{as} \quad R \to \infty.
\]
Therefore \( v \equiv 0 \) and so is \( u \). If \( \alpha = \frac{N-2m}{m} \), then
\[
\int_{B_R} v^p \leq cR^{N-2m-\frac{2m(p+1)}{pq-1}} \leq c.
\]
On the other hand, recall that for \( w > 0, \Delta w \leq 0 \), we have
\[
w(x) \geq c|x|^{2-N} \quad \text{for} \quad |x| \geq 1.
\] (3.6)
Lemma 3.1 implies
\[
\frac{v_{m-1}}{v} \geq cr^2 \frac{u_m}{u}, \quad \frac{u_{m-1}}{u} \geq cr^2 \frac{v_{m-1}}{v}, \quad \frac{u_{k-1}}{u} \geq cr^2 \frac{u_k}{u}, \quad \frac{v_{k-1}}{v} \geq cr^2 \frac{v_k}{v}.
\] (3.7)
(3.6) and (3.7) implies for \( r \geq 1 \),
\[
u \geq cr^{-N+2m}, \quad v \geq cr^{-N+2m}, \quad v \geq cr^{2m} u^\frac{q}{p}
\] (3.8)
If \( q > 1 \), we conclude from (3.8) that
\[
v \geq cr^{2m+(-N+2m)q} \quad \text{for} \quad r \geq 1.
\]
If \( q < 1 \), we apply Lemma 3.2 with \( p_1 = qp \) and \( q_1 = q \). It then follows from Lemma 3.1 that
\[
\left(\frac{u_{m-1}^q}{u_m^q}\right)^\frac{1}{q} \geq cr^2 \left(\frac{v_{m-1}^q}{v_m^q}\right)^\frac{1}{q}, \quad \left(\frac{u_{k-1}^q}{u_k^q}\right)^\frac{1}{q} \geq cr^2 \left(\frac{v_{k-1}^q}{v_k^q}\right)^\frac{1}{q}.
\]
From this and (3.6), we get
\[
\frac{v}{u^q} \geq cr^{-N+2m} \quad \text{for} \quad r \geq 1.
\]
Again we have
\[
v \geq cr^{2m+(-N+2m)q} \quad \text{for} \quad r \geq 1.
\]
Since \( p > 1 \) by our assumption,
\[
\int_{B_R} v^p \geq \omega_N \int_1^R r^{N-1} v^p dx \geq C \int_1^R r^{N-1} v^p dr \geq C \int_1^R r^{N-1+2mp-(N-2m)pq} dr = C \ln R \to \infty.
\]
Contradiction. \( \Box \)

**Proof of Theorem 1.2.** Under the additional assumption \( p \geq 1, q \geq 1, (p, q) \neq (1, 1) \), we conclude from Lemma 2.5 that any positive pair of solutions \((u, v)\) satisfies \((-\Delta)^i u > 0, (-\Delta)^i v > 0, i = 1, 2, \ldots, m-1\). The rest of the proof then follows from the same argument above. \( \Box \)
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References