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A Liouville-type theorem for higher order elliptic systems

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ABSTRACT

We prove there are no positive radial solutions for higher order elliptic system

$$\begin{cases} (-\Delta)^m u = v^p \\ (-\Delta)^m v = u^q \end{cases} \quad \text{in } \mathbb{R}^N$$

if $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N}$. We also show there are no positive solutions to the system under the additional assumption that $\max(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}) \geqslant \frac{N-2m}{m}$. The proof in the radial case uses Rellich's identity and the proof in the general case relies on growth estimates of the spherical average of the solution.

1. Introduction

In this paper, we consider positive solutions ($u > 0$, $v > 0$) of the following higher order elliptic system

$$\begin{cases} (-\Delta)^m u = v^p \\ (-\Delta)^m v = u^q \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 0$, $q > 0$ and $N \geqslant 3$. We are mainly concerned with the question of nonexistence of such positive solutions.

When $m = 1$, (1.1) becomes Lane–Emden system

$$\begin{cases} \Delta u + v^p = 0 \\ \Delta v + u^q = 0 \end{cases} \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

It has been conjectured that the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ is the dividing curve for existence and nonexistence of positive solutions of (1.2). The conjecture was completely solved in the case of radial solutions [4,7,9]. Mitidieri [4] showed that there is no positive radial solutions to (1.2) below the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ if $p > 1$, $q > 1$; the condition $p > 1$, $q > 1$ was later relaxed to $p > 0$, $q > 0$ by Serrin and Zou [7,9]. Furthermore, it is proved by Serrin and Zou [9] that there are infinitely many positive radial solutions above the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. Therefore $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ serves as the dividing curve for existence and nonexistence of positive radial solutions of (1.2).

The question for the general positive solution to (1.2), to the best of our knowledge, has not been completely solved yet. Partial answers have been given over the years. Souto [11] proved nonexistence of positive C^2 solutions below curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N-1}$ when $p, q > 0$. Felmer and de Figueiredo [2] showed that when $0 < p, q \leqslant \frac{N+2}{N-2}$ and $(p, q) \neq (\frac{N+2}{N-2}, \frac{N+2}{N-2})$,

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(1.2) has no positive C^2 solutions. Further evidence supporting the conjecture can be found in [5], where it is shown that there exists no positive supersolutions to (1.2) below the curve

$$\left\{p > 0,\ q > 0\colon\ \frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N-2}\max\left(\frac{1}{p+1}, \frac{1}{q+1}\right)\right\}. \tag{1.3}$$

We refer to (1.3) as S curve and the hyperbola in the conjecture $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ will be referred as Sobolev's hyperbola throughout the paper. For $0 < p, q$, if $pq \leqslant 1$ or $pq > 1$ and $\max(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}) \geqslant N-2$, nonexistence of positive solutions was proved by Serrin and Zou in [8]. Direct calculation shows this is the same range of (p, q) as region below and on S curve. Furthermore, Serrin and Zou [8] showed (1.2) admits no positive solutions satisfying algebraic growth at infinity below the Sobolev hyperbola when $N = 3$. For the special case $\min(p, q) = 1$, the conjecture was proved by C.-S. Lin [3]. Busca and Manásevich [1] proved that if $p, q > 0$, $pq > 1$,

$$\frac{N-2}{2} \leqslant \min\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \leqslant \max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) < N-2,$$

and

$$\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \neq \left(\frac{N-2}{2}, \frac{N-2}{2}\right),$$

there is no positive classical solutions to (1.2). Most recently, the conjecture was fully solved in the case $N = 3$ by Poláčik, Quittner and Souplet [6] and by Souplet [10] when $N = 4$. Souplet also proved the conjecture when $N \geqslant 5$ under the additional assumption that $\max(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}) > N-3$.

Comparing to the Lane–Emden system, less is known about the higher order system (1.1). In the single equation case, Mitidieri [4] proved that for $1 < q < \frac{N+4m}{N-4m}$, $N > 4m$, the problem

$$\begin{cases} \Delta^{2m} u = u^q, & \\ (-\Delta)^s u \geqslant 0, \quad s = 1, 2, \ldots, 2m-1 & \end{cases} \quad \text{in } \mathbb{R}^N$$

has no nontrivial positive radial solution of class $C^{4m}(\mathbb{R}^N)$. In this paper, we prove the following generalization of the Liouville-type theorem to higher order elliptic system. Our first Liouville-type theorem deals with radially symmetric positive solutions of (1.1).

Theorem 1.1. *If $N \geqslant 3$, $N > 2m$, $p \geqslant 1$, $q \geqslant 1$, $(p, q) \neq (1, 1)$ and $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2m}{N}$, the problem* (1.1) *has no nontrivial positive radial solutions of class $C^{2m}(\mathbb{R}^N)$.*

Our second theorem handles Liouville properties of general solutions of (1.1).

Theorem 1.2. *$N \geqslant 3$, $N > 2m$, if $p \geqslant 1$, $q \geqslant 1$, $(p, q) \neq (1, 1)$ and*

$$\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) \geqslant N - 2m,$$

the problem (1.1) *has no nontrivial positive solutions of class $C^{2m}(\mathbb{R}^N)$.*

The assumption $p \geqslant 1$, $q \geqslant 1$, $(p, q) \neq (1, 1)$ in the previous two theorems is only needed to show that any positive solution (u, v) of (1.1) satisfies

$$(-\Delta)^i u > 0, \qquad (-\Delta)^i v > 0, \quad i = 1, 2, \ldots, m-1.$$

We shall prove the following version of the radial and general case and Theorem 1.1 and Theorem 1.2 are obtained as corollaries of the following theorems respectively.

Theorem 1.3. *If $N \geqslant 3$, $N > 2m$, and $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2m}{N}$, then the problem*

$$\begin{cases} (-\Delta)^m u = v^p & \\ (-\Delta)^m v = u^q & \text{in } \mathbb{R}^N, \\ (-\Delta)^i u > 0, \quad (-\Delta)^i v > 0, \quad i = 1, 2, \ldots, m-1 & \end{cases}$$

has no nontrivial positive radial solutions of class $C^{2m}(\mathbb{R}^N)$.

Theorem 1.4. *Let $N \geqslant 3$, $N > 2m$, then the problem*

$$\begin{cases} (-\Delta)^m u = v^p \\ (-\Delta)^m v = u^q & \text{in } \mathbb{R}^N, \\ (-\Delta)^i u > 0, \quad (-\Delta)^i v > 0, \quad i = 1, 2, \ldots, m-1 \end{cases} \tag{1.4}$$

has no nontrivial positive solutions of class $C^{2m}(\mathbb{R}^N)$ if $pq \leqslant 1$ or if $pq > 1$ and $\max(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}) \geqslant N - 2m$.

The paper is organized as follows. Section 2 presents proof of radial case (Theorem 1.1 and Theorem 1.3), Section 3 is devoted to the proof of general case (Theorem 1.2 and Theorem 1.4). The proof of the radial case uses Rellich's Identity and the proof of the general solution case relies on growth estimates of the spherical average of the solution.

2. Radial solutions

First we recall the following function defined in [4]

$$R_n(u, v) = \int_\Omega \Delta^n u(x, \nabla v) + \Delta^n v(x, \nabla u)\, dx$$

where $u, v \in C^{2n}(\overline{\Omega})$, $n \geqslant 1$. If $n = 1$, we have

$$R_1(u, v) = \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial n}(x, \nabla v) + \frac{\partial v}{\partial n}(x, \nabla u) - (\nabla u, \nabla v)(x, n) \right\} ds + (N-2) \int_\Omega (\nabla u, \nabla v)\, dx.$$

If $n = 2$,

$$R_2(u, v) = R_1(\Delta u, v) + R_1(u, \Delta v) - B(u, v) \tag{2.1}$$

where

$$B(u, v) = \int_{\partial\Omega} \Delta u \Delta v(x, n)\, ds - N \int_\Omega \Delta u \Delta v\, dx. \tag{2.2}$$

We quote the following lemma from [4]

Lemma 2.1. *(See [4, Lemma 2.2].) If $u, v \in C^{2n}(\overline{\Omega})$, then for $1 \leqslant s \leqslant n-2$*

$$R_n(u, v) = \sum_{k=0}^{s} R_{n-s}\big(\Delta^k u, \Delta^{s-k} v\big) - \sum_{k=0}^{s-1} R_{n-(s+1)}\big(\Delta^{k+1} u, \Delta^{s-k} v\big). \tag{2.3}$$

Remark 2.2. An immediate consequence of Lemma 2.1 is the following implicit form of Rellich's identity. If $u, v \in C^{2n}(\overline{\Omega})$, then

$$R_n(u, v) = \sum_{k=0}^{n-1} R_1\big(\Delta^k u, \Delta^{n-1-k} v\big) - \sum_{k=0}^{n-2} B\big(\Delta^k u, \Delta^{n-2-k} v\big). \tag{2.4}$$

Proof. Choose $s = n-2$ in (2.3), taking into account of (2.1) and (2.2), (2.4) follows. □

Lemma 2.3. *(See [4, Lemma 3.1].) If $\psi \in C^2(\mathbb{R}^N)$ $(N \geqslant 3)$ is positive, radial and superharmonic (i.e. $\Delta\psi \leqslant 0$ in $\mathbb{R}^N$), then*

$$r\psi'(r) + (N-2)\psi(r) \geqslant 0 \quad \text{for } r > 0.$$

For $w \in C(\mathbb{R}^N)$, denote the spherical average of w by

$$\overline{w}(r) = \frac{1}{\omega_n} \int_{S^{N-1}} w(r, \theta), \quad r > 0,$$

where (r, θ) are spherical coordinates and $\omega_N = |S^{N-1}|$ is the area of the unit sphere in $\mathbb{R}^N$. We quote the following lemma from [8].

Lemma 2.4. *(See [8, Lemma 2.3].) Suppose that $w \geqslant 0$ is nontrivial and satisfies*

$$\Delta w \leqslant 0, \quad x \in \mathbb{R}^N.$$

Then for $a \in (0,1]$, $(\overline{w^a})' \leqslant 0$. In particular, $\overline{w^a}$ is a nonincreasing function of r.

Lemma 2.5. *Let (u, v) be a pair of positive solutions of (1.1), if $p \geqslant 1$, $q \geqslant 1$, $(p,q) \neq (1,1)$, then we have*

$$(-\Delta)^i u > 0, \qquad (-\Delta)^i v > 0, \quad i = 1, 2, \ldots, m-1.$$

Proof. We follow an idea in [12]. Let $u_i = (-\Delta)^i u$, $v_i = (-\Delta)^i v$, $i = 0, 1, \ldots, m-1$ with $u_0 = u$, $v_0 = v$. We first prove $v_{m-1} > 0$. Suppose not, there exists $x_0 \in \mathbb{R}^n$ such that

$$v_{m-1}(x_0) < 0.$$

Without loss of generality, we assume $x_0 = 0$. Then we have, since $p \geqslant 1$, $q \geqslant 1$,

$$\Delta \bar{u} + \overline{u_1} = 0,$$
$$\Delta \overline{u_1} + \overline{u_2} = 0,$$
$$\ldots$$
$$\Delta \overline{u_{m-1}} + \bar{v}^p \leqslant 0,$$

and

$$\Delta \bar{v} + \overline{v_1} = 0,$$
$$\Delta \overline{v_1} + \overline{v_2} = 0,$$
$$\ldots$$
$$\Delta \overline{v_{m-1}} + \bar{u}^q \leqslant 0.$$

Since $\overline{v_{m-1}}(0) < 0$ and $\overline{v_{m-1}}' < 0$, we have

$$\overline{v_{m-1}}(r) < 0 \quad \text{for all } r > \overline{r_1} = 0.$$

It then follows easily that

$$\overline{v_{m-2}}' > -\frac{\overline{v_{m-1}}(0)}{n} r.$$

hence

$$\overline{v_{m-2}}(r) \geqslant c_2 r^2, \quad \text{for } r \geqslant \overline{r_2} > \overline{r_1}.$$

The same arguments show that

$$\overline{v_{m-3}}(r) \leqslant -c_3 r^4, \quad \text{for } r \geqslant \overline{r_3} > \overline{r_2}.$$

and

$$(-1)^i \overline{v_{m-i}}(r) \geqslant c_i r^{2(i-1)}, \quad \text{for } r \geqslant \overline{r_i},\ i = 1, \ldots, m.$$

Hence if m is odd, we have a contradiction with the fact that $v > 0$.

So m must be even and we have

$$\bar{v} \geqslant c_0 r^{\sigma_0}, \qquad \sigma_0 = 2(m-1)$$

and

$$(-1)^i \overline{v_{m-i}} > 0$$

for $r > \overline{r_0} > 0$.

Setting $A = 2^{q+1}(n + 2m + 2mq + 2pq(m-1))^{1+q}$ and suppose that

$$\bar{v}(r) \geqslant \frac{c_0^{(pq)^k} r^{\sigma_k}}{A^{b_k}}, \quad \text{for } r \geqslant r_k.$$

Then we have

$$r^{n-1} \overline{u_{m-1}}'(r) \leqslant r_k^{n-1} \overline{u_{m-1}}'(r_k) - \int_{r_k}^{r} s^{n-1} \bar{v}^p(s)\, ds,$$

therefore

$$\overline{u_{m-1}}'(r) \leqslant -\frac{r^{1+p\sigma_k} - r_k^{1+p\sigma_k}}{A^{pb_k}(p\sigma_k + n)} c_0^{q^k p^{k+1}},$$

hence

$$\overline{u_{m-1}}'(r) \leqslant -\frac{r^{1+p\sigma_k}}{2A^{pb_k}(p\sigma_k + n)} c_0^{q^k p^{k+1}}$$

for $r \geqslant 2^{\frac{1}{p\sigma_k+1}} r_k$. Similarly

$$\overline{u_{m-1}}(r) \leqslant -\frac{r^{2+p\sigma_k}}{4A^{pb_k}(p\sigma_k + n)(p\sigma_k + 2)} c_0^{q^k p^{k+1}}$$

for $r \geqslant 2^{\frac{1}{1+p\sigma_k}} 2^{\frac{1}{2+p\sigma_k}} r_k$.

Hence

$$\overline{u_{m-1}}(r) \leqslant -\frac{r^{2+p\sigma_k}}{4A^{pb_k}(p\sigma_k + n)^2} c_0^{q^k p^{k+1}}$$

for $r \geqslant 2^{\frac{2}{1+p\sigma_k}} r_k$.

By induction, we have

$$(-1)^i \overline{u_{m-i}}(r) \geqslant \frac{c_0^{q^k p^{k+1}} r^{2i+p\sigma_k}}{(n + 2m + p\sigma_k)^{2i} A^{pb_k} 4^i} \quad \text{for } r \geqslant 2^{\frac{2i}{1+p\sigma_k}} r_k.$$

In particular,

$$\bar{u}(r) \geqslant \frac{c_0^{q^k p^{k+1}} r^{2m+p\sigma_k}}{(n + 2m + p\sigma_k)^{2m} A^{pb_k} 4^m} \quad \text{for } r \geqslant 2^{\frac{2m}{1+p\sigma_k}} r_k = s_k.$$

Since

$$r^{n-1} \overline{v_{m-1}}'(r) \leqslant s_k^{n-1} \overline{v_{m-1}}'(s_k) - \int_{s_k}^{r} s^{n-1} \bar{u}^q(s)\, ds,$$

which implies

$$\overline{v_{m-1}}'(r) \leqslant -\frac{r^{1+2mq+pq\sigma_k} - s_k^{1+2mq+pq\sigma_k}}{(n + 2m + p\sigma_k)^{2mq}(n + 2mq + pq\sigma_k) A^{pqb_k} 4^{mq}} c_0^{(qp)^{k+1}},$$

hence

$$\overline{v_{m-1}}'(r) \leqslant -\frac{r^{1+2mq+pq\sigma_k}}{2 \cdot 4^{mq} A^{pqb_k}(n + 2m + p\sigma_k)^{2mq}(n + 2mq + pq\sigma_k)} c_0^{(qp)^{k+1}}$$

for $r \geqslant 2^{\frac{1}{pq\sigma_k+2mq+1}} s_k$. Similarly

$$\overline{v_{m-1}}(r) \leqslant -\frac{r^{2+2mq+pq\sigma_k}}{4^{mq+1} A^{pqb_k}(n + 2m + p\sigma_k)^{2mq}(n + 2mq + pq\sigma_k)(pq\sigma_k + 2mq + 2)} c_0^{(qp)^{k+1}}$$

for $r \geqslant 2^{\frac{1}{1+2mq+pq\sigma_k}} 2^{\frac{1}{2+2mq+pq\sigma_k}} s_k$.

Hence

$$\overline{v_{m-1}}(r) \leqslant -\frac{r^{2+2mq+pq\sigma_k}}{4^{mq+1} A^{pqb_k}(n + 2m + p\sigma_k)^{2mq}(pq\sigma_k + n + 2mq)^2} c_0^{(qp)^{k+1}}$$

for $r \geqslant 2^{\frac{2}{1+pq\sigma_k+2mq}} s_k$.

By induction, we have

$$(-1)^i \overline{v_{m-i}}(r) \geqslant \frac{c_0^{(qp)^{k+1}} r^{2i+2mq+pq\sigma_k}}{(n + 2m + 2mq + pq\sigma_k)^{2i}(n + 2m + p\sigma_k)^{2mq} A^{pqb_k} 4^{i+mq}}$$

for $r \geqslant 2^{\frac{2i}{1+2mq+pq\sigma_k}} s_k$. Hence

$$\bar{v}(r) \geqslant \frac{c_0^{(qp)^{k+1}} r^{2m+2mq+pq\sigma_k}}{(n+2m+2mq+pq\sigma_k)^{2m}(n+2m+p\sigma_k)^{2mq} A^{pqb_k} 4^{m+mq}} \tag{2.5}$$

for $r \geqslant 2^{\frac{2m}{1+2mq+pq\sigma_k}} s_k$. Set

$$\begin{aligned} &\sigma_0 = 2(m-1), \qquad r_0 = \overline{r_0}, \\ &\sigma_{k+1} = 2m + 2mq + pq\sigma_k, \\ &r_{k+1} = 2^{\frac{2m}{1+2mq+pq\sigma_k}} 2^{\frac{2m}{1+p\sigma_k}} r_k. \end{aligned} \tag{2.6}$$

By our choice of A, one can show via induction that

$$(n+2m+2mq+pq\sigma_k)^{2m}(n+2m+p\sigma_k)^{2mq} 4^{m+mq} \leqslant A^{2m(k+1)}.$$

We can then set

$$b_0 = 0, \quad b_{k+1} = pqb_k + 2m(k+1), \tag{2.7}$$

and rewrite (2.5) as

$$\bar{v}(r) \geqslant \frac{c_0^{(qp)^{k+1}}}{A^{b_{k+1}}} r^{\sigma_{k+1}}, \quad r \geqslant r_{k+1}.$$

Notice that

$$r_{k+1} \leqslant c r_0,$$

where

$$c = 2^{\sum_{k=0}^{\infty} \frac{2m}{1+2mq+pq\sigma_k} + \frac{2m}{1+p\sigma_k}}.$$

From iteration formula (2.6), (2.7), we have

$$\begin{aligned} &\sigma_k = 2(m-1)(pq)^k + 2m(q+1)\frac{(pq)^k - 1}{pq-1}, \\ &b_k = 2m\frac{(pq)^{k+1} - (k+1)pq + k}{(pq-1)^2}. \end{aligned}$$

Take $M > 1$ large enough so that $MA^{\frac{2}{pq-1}} \geqslant 2cr_0$ if $c_0 \geqslant 1$ and $MA^{\frac{2}{pq-1}} c_0^{-1} \geqslant 2cr_0$ if $c_0 < 1$, and take $r_1 = MA^{\frac{2}{pq-1}}$ or $MA^{\frac{2}{pq-1}} c_0^{-1}$ depending on whether c_0 is greater or less than 1, then we have

$$\bar{v}(r_1) \geqslant \left[A^{\frac{2}{pq-1}}\right]^{(6m-4+4mq)(pq)^{k+1} - 4(m+mq) + 2m(k+1)pq - 2mk} \to \infty \quad \text{as } k \to \infty,$$

contradiction to the fact that r_1 is independent of k.

Hence $v_{m-1} > 0$. $u_{m-1} > 0$ can be proved similarly. Next we claim $v_{m-i} \geqslant 0$, $u_{m-i} \geqslant 0$ for $i = 2, \ldots, m-1$. Proof is exactly the same except that we need to take extra care when m is odd. We omit the details. $\square$

Lemma 2.6. *If $pq > 1$, $p \geqslant 1$, $q \geqslant 1$, $(u, v) \in C^{2n}(\mathbb{R}^N)$, $N \geqslant 3$, is a pair of positive radial solution of* (1.1), *then*

$$u(r) \leqslant Cr^{-\frac{2m(p+1)}{pq-1}}, \qquad v(r) \leqslant Cr^{-\frac{2m(q+1)}{pq-1}} \quad r > 0,$$

and for $k = 0, 1, \ldots, m-1$, $r > 0$

$$\begin{aligned} &\left|r^N \Delta^{k+1} u \Delta^{m-1-k} v\right| \leqslant Cr^{N-2m-\frac{2m(p+q+2)}{pq-1}}, \\ &\left|r^{N-1} \left(\Delta^k u\right)' \Delta^{m-1-k} v\right| \leqslant Cr^{N-2m-\frac{2m(p+q+2)}{pq-1}}, \\ &\left|r^{N-1} \Delta^k u \left(\Delta^{m-1-k} v\right)'\right| \leqslant Cr^{N-2m-\frac{2m(p+q+2)}{pq-1}}, \\ &\left|r^N \left(\Delta^k u\right)' \left(\Delta^{m-1-k} v\right)'\right| \leqslant Cr^{N-2m-\frac{2m(p+q+2)}{pq-1}}. \end{aligned} \tag{2.8}$$

Proof. Let $u_k = (-\Delta)^k u$, $v_k = (-\Delta)^k v$. By Lemma 2.5, $u_k > 0$, $v_k > 0$ and $-\Delta u_k > 0$, $-\Delta v_k > 0$, Lemma 2.4 implies $u_k' < 0$, $v_k' < 0$. And

$$-\left(u_k' r^{N-1}\right)' = u_{k+1} r^{N-1}, \qquad -\left(v_k' r^{N-1}\right)' = v_{k+1} r^{N-1}, \quad r > 0. \tag{2.9}$$

Integrating (2.9), and using the fact that $u_k' < 0$, $v_k' < 0$, we obtain

$$-u_k' r^{N-1} = \frac{r^N}{N} u_{k+1} - \frac{1}{N} \int_0^r s^N u_{k+1}' \, ds \geqslant \frac{r^N}{N} u_{k+1}, \tag{2.10}$$

$$-v_k' r^{N-1} = \frac{r^N}{N} v_{k+1} - \frac{1}{N} \int_0^r s^N v_{k+1}' \, ds \geqslant \frac{r^N}{N} v_{k+1}. \tag{2.11}$$

An application of Lemma 2.3 to $u_{k,} v_k$ yields

$$r u_k' + (N-2) u_k \geqslant 0, \tag{2.12}$$

$$r v_k' + (N-2) v_k \geqslant 0. \tag{2.13}$$

It follows from (2.10) and (2.12) that

$$r^2 u_{k+1} \leqslant N(N-2) u_k, \quad k = 0, 1, \ldots, m-1, \tag{2.14}$$

and from (2.11), (2.13) that

$$r^2 v_{k+1} \leqslant N(N-2) v_k, \quad k = 0, 1, \ldots, m-1. \tag{2.15}$$

(2.14) and (2.15) imply

$$r^{2m} v^p = r^{2m} u_m \leqslant N^m (N-2)^m u,$$
$$r^{2m} u^q = r^{2m} v_m \leqslant N^m (N-2)^m v,$$

from which we can easily solve if $pq > 1$ that

$$u \leqslant C r^{\frac{-2m(p+1)}{pq-1}}, \qquad v \leqslant C r^{\frac{-2m(q+1)}{pq-1}},$$

and for $k = 1, 2, \ldots, m-1$

$$u_k \leqslant C u r^{-2k} = C r^{-\frac{2m(p+1)}{pq-1} - 2k}, \qquad v_k \leqslant C v r^{-2k} = c r^{-\frac{2m(q+1)}{pq-1} - 2k};$$
$$0 < -u_k' < C r^{-\frac{2m(p+1)}{pq-1} - 2k - 1}, \qquad 0 < -v_k' < C r^{-\frac{2m(q+1)}{pq-1} - 2k - 1}.$$

(2.8) follows. □

Proof of Theorem 1.1. We prove by contradiction. Let (u, v) be a pair of nontrivial positive radial solution of (1.1), we have

$$-\left(u_{m-1}' r^{N-1}\right)' = v^p r^{N-1}, \tag{2.16}$$

$$-\left(v_{m-1}' r^{N-1}\right)' = u^q r^{N-1}. \tag{2.17}$$

Multiply (2.16) by v and (2.17) by u and integrating by parts on $(0, r)$ we obtain

$$-u_{m-1}'(r) v(r) r^{N-1} + \int_0^r u_{m-1}'(s) v'(s) s^{N-1} \, ds = \int_0^r v^{p+1}(s) s^{N-1} \, ds, \tag{2.18}$$

$$-v_{m-1}'(r) u(r) r^{N-1} + \int_0^r v_{m-1}'(s) u'(s) s^{N-1} \, ds = \int_0^r u^{q+1}(s) s^{N-1} \, ds. \tag{2.19}$$

Using the fact that

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N} \quad \Longleftrightarrow \quad \frac{N-2m}{2m} < \frac{p+q+2}{pq-1} \tag{2.20}$$

and (2.8), we conclude

$$u_{m-1}'(r) v(r) r^{N-1} \to 0, \qquad v_{m-1}'(r) u(r) r^{N-1} \to 0,$$

therefore

$$\int_0^\infty u'_{m-1}(s)v'(s)s^{N-1}\,ds=\int_0^\infty v^{p+1}(s)s^{N-1}\,ds,$$
$$\int_0^\infty v'_{m-1}(s)u'(s)s^{N-1}\,ds=\int_0^\infty u^{q+1}(s)s^{N-1}\,ds. \tag{2.21}$$

On the other hand

$$\begin{aligned}
&\frac{N}{p+1}\int_0^r v^{p+1}(s)s^{N-1}\,ds+\frac{N}{q+1}\int_0^r u^{q+1}(s)s^{N-1}\,ds\\
&\quad=\frac{1}{p+1}v^{p+1}(r)r^N+\frac{1}{q+1}u^{q+1}(r)r^N-\int_0^r v^p(s)v'(s)s^N\,ds-\int_0^r u^q(s)u'(s)s^N\,ds\\
&\quad=\frac{1}{p+1}v^{p+1}(r)r^N+\frac{1}{q+1}u^{q+1}(r)r^N-\frac{(-1)^m}{\omega_N}R_m(u,v).
\end{aligned}\tag{2.22}$$

Here we used

$$\begin{aligned}
(-1)^m R_m(u,v)&=\int_{B_r}(-\Delta)^m u(x)(x,\nabla v)+(-\Delta)^m v(x,\nabla u)\,dx\\
&=\int_{B_r}v^p(x)(x,\nabla v)+u^q(x)(x,\nabla u)\,dx\\
&=\omega_N\int_0^r v^p(s)v'(s)s^N+u^q(s)u'(s)s^N\,ds.
\end{aligned}$$

By (2.4), we have

$$\begin{aligned}
R_m(u,v)&=\sum_{k=0}^{m-1}R_1\big(\Delta^k u,\Delta^{m-1-k}v\big)-\sum_{k=0}^{m-2}B\big(\Delta^k u,\Delta^{m-2-k}v\big)\\
&=\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\big(x,\nabla\Delta^{m-1-k}v\big)+\frac{\partial\Delta^{m-1-k}v}{\partial n}\big(x,\nabla\Delta^k u\big)\,dx\\
&\quad-\sum_{k=0}^{m-1}\int_{\partial B_r}\big(\nabla\Delta^k u,\nabla\Delta^{m-1-k}v\big)(x,n)\,ds+(N-2)\sum_{k=0}^{m-1}\int_{B_r}\big(\nabla\Delta^k u,\nabla\Delta^{m-1-k}v\big)\,dx\\
&\quad-\sum_{k=0}^{m-2}\int_{\partial B_r}\big(\Delta^{k+1}u,\Delta^{m-1-k}v\big)(x,n)\,ds+N\sum_{k=0}^{m-2}\int_{B_r}\big(\Delta^{k+1}u,\Delta^{m-1-k}v\big)\,dx.
\end{aligned}\tag{2.23}$$

Since

$$\int_{B_r}\big(\nabla\Delta^k u,\nabla\Delta^{m-1-k}v\big)\,dx=\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v\,ds-\int_{B_r}\big(\Delta^{k+1}u,\Delta^{m-1-k}v\big)\,dx,$$

we can rewrite (2.23) as

$$\begin{aligned}
R_m(u,v)&=\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\big(x,\nabla\Delta^{m-1-k}v\big)+\frac{\partial\Delta^{m-1-k}v}{\partial n}\big(x,\nabla\Delta^k u\big)\,ds-\sum_{k=0}^{m-1}\int_{\partial B_r}\big(\nabla\Delta^k u,\nabla\Delta^{m-1-k}v\big)(x,n)\,ds\\
&\quad-\sum_{k=0}^{m-2}\int_{\partial B_r}\big(\Delta^{k+1}u,\Delta^{m-1-k}v\big)(x,n)\,ds+(N-2)\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v\,ds
\end{aligned}$$

$$
\begin{aligned}
&- (N-2)\sum_{k=0}^{m-1}\int_{B_r}\big(\Delta^{k+1}u, \Delta^{m-1-k}v\big)\,dx + N\sum_{k=0}^{m-2}\int_{B_r}\big(\Delta^{k+1}u, \Delta^{m-1-k}v\big)\,dx\\
&= \sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\big(x, \nabla\Delta^{m-1-k}v\big) + \frac{\partial\Delta^{m-1-k}v}{\partial n}\big(x, \nabla\Delta^k u\big)\,ds\\
&- \sum_{k=0}^{m-1}\int_{\partial B_r}\big(\nabla\Delta^k u, \nabla\Delta^{m-1-k}v\big)(x,n)\,ds\\
&- \sum_{k=0}^{m-2}\int_{\partial B_r}\big(\Delta^{k+1}u, \Delta^{m-1-k}v\big)(x,n)\,ds + (N-2)\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v\,ds\\
&- (N-2)\int_{B_r}\big(\Delta^m u, v\big)\,dx + 2\sum_{k=0}^{m-2}\int_{B_r}\big(\Delta^{k+1}u, \Delta^{m-1-k}v\big)\,dx.
\end{aligned}
\tag{2.24}
$$

Recall

$$
\int_{B_r}\big(\Delta^{k+1}u, \Delta^{m-1-k}v\big)\,dx - \int_{B_r}\big(\Delta^k u, \Delta^{m-k}v\big)\,dx = \int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v - \frac{\partial\Delta^{m-1-k}v}{\partial n}\Delta^k u
\tag{2.25}
$$

for any k, letting $r\to\infty$ in (2.22), it follows from Lemma 2.6, (2.20), (2.24) and (2.25) that

$$
\frac{N}{p+1}\int_0^\infty v^{p+1}(s)s^{N-1}\,ds + \frac{N}{q+1}\int_0^\infty v^{p+1}(s)s^{N-1}\,ds = (N-2m)\int_0^\infty v^{p+1}(s)s^{N-1}\,ds
$$

which is a contradiction to $\frac{N}{p+1}+\frac{N}{q+1} > N-2m$. $\square$

Proof of Theorem 1.3. Notice the only place where assumption $p\geqslant 1$, $q\geqslant 1$, $(p,q)\neq(1,1)$ is needed in the proof of Theorem 1.1 is to show $u_k=(-\Delta)^k u>0$, $v_k=(-\Delta)^k v>0$ for any given pair of positive solution (u,v). Theorem 1.3 follows from the same argument above. $\square$

3. General solutions

When $pq>1$, we introduce the following notation

$$
\alpha = \frac{2(p+1)}{pq-1}, \qquad \beta = \frac{2(q+1)}{pq-1}
$$

and assume $\alpha\geqslant\beta$ throughout the rest of the section. The assumption

$$
\frac{1}{p+1}+\frac{1}{q+1} > \frac{N-2m}{m}
$$

can be rewritten as

$$
m\alpha + m\beta > N-2m.
$$

We first quote the following lemmas from [8].

Lemma 3.1. *(See [8, Lemma 2.7].) Suppose $z=z(r)>0$ satisfies*

$$
z'' + \frac{n-1}{r}z' + \phi(r) \leqslant 0, \quad r>0
$$

with $\phi(r)$ non-negative and nonincreasing, and z' bounded for r near 0. Then

$$
z(r) \geqslant cr^2\phi(r)
$$

where $c=c(N)$.

Lemma 3.2. *Let u be a positive solution*

$$\Delta u + v^p = 0,$$

where v is a positive function, then

$$U_1'' + \frac{N-1}{r}U_1' + V_1^p \leqslant 0 \quad \text{for } 0 < p_1 \leqslant p$$

where

$$U_1(r) = \left(\overline{u^{\frac{p_1}{p}}}\right)^{\frac{p}{p_1}}, \qquad V_1(r) = \left(\overline{v^{p_1}}\right)^{\frac{1}{p_1}}.$$

Proof. See proof of Lemma 2.6 in [8]. $\square$

We can now prove the following growth estimate on $\bar{u}, \bar{v}$.

Lemma 3.3. *If $pq = 1$, there is no nontrivial positive solution of* (1.4). *If (u, v) is a positive solution of* (1.4) *and $pq > 1$, there exists a positive constant $M = M(p, q, n)$ such that*

$$\bar{u}(r) \leqslant Mr^{-\frac{2m(p+1)}{pq-1}}, \qquad \bar{v}(r) \leqslant Mr^{-\frac{2m(q+1)}{pq-1}} \quad \text{for } r > 0, \tag{3.1}$$

and for $k = 1, \dots, m-1$, $u_k = (-\Delta)^k u$, $v_k = (-\Delta)^k v$

$$\overline{u_k}(r) \leqslant Mr^{-\frac{2m(p+1)}{pq-1}-2k}, \qquad \overline{v_k}(r) \leqslant Mr^{-\frac{2m(q+1)}{pq-1}-2k} \quad \text{for } r > 0. \tag{3.2}$$

Proof. We first consider the case $p, q \geqslant 1$. Taking the spherical average of (1.4) and using Jensen's inequality, we have

$$\overline{u_{m-1}}'' + \frac{N-1}{r}\overline{u_{m-1}}' + \bar{v}^p \leqslant 0,$$
$$\overline{v_{m-1}}'' + \frac{N-1}{r}\overline{v_{m-1}}' + \bar{u}^q \leqslant 0,$$

and

$$\overline{u_{k-1}}'' + \frac{N-1}{r}\overline{u_{k-1}}' + \overline{u_k} = 0,$$
$$\overline{v_{k-1}}'' + \frac{N-1}{r}\overline{v_{k-1}}' + \overline{v_k} = 0$$

for $k = 1, 2, \dots, m-1$. By assumption, $\overline{u_k}, \overline{v_k}$ are nonincreasing for $k = 0, 1, \dots, m-1$, therefore we get from Lemma 3.1

$$\begin{aligned} &\overline{u_{m-1}} \geqslant cr^2\bar{v}^p, \qquad \overline{v_{m-1}} \geqslant cr^2\bar{u}^q, \\ &\overline{u_{k-1}} \geqslant cr^2\overline{u_k}, \qquad \overline{v_{k-1}} \geqslant cr^2\overline{v_k}. \end{aligned} \tag{3.3}$$

Solving (3.3), we obtain (3.1) and (3.2). In particular, we immediately get a contradiction if $pq = 1$.

Next we assume $pq \geqslant 1$ and $q < 1$. Taking spherical average of (1.1), we have

$$\overline{v_{m-1}}'' + \frac{N-1}{r}\overline{v_{m-1}}' + \overline{u^q} = 0.$$

From Lemma 3.1, we obtain

$$\begin{aligned} &\overline{u_{m-1}} \geqslant cr^2\bar{v}^p, \qquad \overline{v_{m-1}} \geqslant cr^2\overline{u^q}, \\ &\overline{u_{k-1}} \geqslant cr^2\overline{u_k}, \qquad \overline{v_{k-1}} \geqslant cr^2\overline{v_k}. \end{aligned} \tag{3.4}$$

Solving (3.4), taking into account of Lemma 3.2 and that $(\overline{u^{\frac{1}{p}}})^p \leqslant (\overline{u^q})^{\frac{1}{q}}$ since $q \geqslant \frac{1}{p}$. We get a contradiction when $pq = 1$ and if $pq > 1$ it follows that

$$\overline{u^q} \leqslant cr^{-m\alpha q}, \qquad \bar{v} \leqslant cr^{-m\beta}.$$

We can now follow the same argument in p. 644 of [8] to conclude (3.1). (3.2) then follows from (3.1) and (3.4). $\square$

Lemma 3.4. *Suppose that $pq > 1$ and (u, v) is a positive solution of* (1.4). *Then*

$$\int_{B_R} u^q \leqslant cR^{N-2m-\frac{2m(q+1)}{pq-1}}, \qquad \int_{B_R} v^p \leqslant cR^{N-2m-\frac{2m(p+1)}{pq-1}}, \tag{3.5}$$

where $c = c(p, q, n)$.

Proof.

$$\begin{aligned}
\int_{B_R} u^q &\leqslant cR^{N-2}\int_R^{2R} r^{1-N}\int_{B_r} u^q = cR^{n-2}\int_R^{2R} r^{1-N}\int_{B_r} v_m \\
&= -cR^{N-2}\int_R^{2R} r^{1-N}\int_{\partial B_r}\frac{\partial v_{m-1}}{\partial n} \\
&= -cR^{N-2}\int_R^{2R} \overline{v_{m-1}}' \leqslant cR^{N-2}\overline{v_{m-1}}(R) \\
&\leqslant MR^{N-2-\frac{2m(q+1)}{pq-1}-2(m-1)} \\
&= cR^{N-2m-\frac{2m(q+1)}{pq-1}}
\end{aligned}$$

the second inequality can be proved similarly. □

Lemma 3.5. *Suppose $pq \leqslant 1$. Then* (1.4) *has no nontrivial and non-negative solutions.*

Proof. The case $pq = 1$ is proved in Lemma 3.3. We assume $pq < 1$. Choose s and θ such that

$$p < s < \frac{1}{q}, \qquad 0 < \theta < \min\left(1, \frac{1}{s}\right),$$

and put $\gamma = \theta s$. It follows that

$$\theta p < \gamma < 1, \qquad \gamma q < \theta < 1$$

and

$$a = \frac{1-\theta}{1-\gamma q} \in (0,1), \qquad b = \frac{1-\gamma}{1-\theta p} \in (0,1).$$

By Hölder's inequality,

$$\overline{u^\theta} \leqslant \left(\overline{u^{\gamma q}}\right)^a \bar{u}^{1-a} \leqslant c\left(\overline{u^{\gamma q}}\right)^a,$$

since $\bar{u}$ is decreasing thus bounded. Similarly,

$$\overline{v^\gamma} \leqslant c\left(\overline{v^{\theta p}}\right)^b.$$

On the other hand, apply Lemma 3.2 with $p_1 = \theta p$, $q_1 = \gamma q$, we conclude from Lemma 3.1 that

$$U_{m-1} = \left(\overline{u^\theta_{m-1}}\right)^{\frac{1}{\theta}} \geqslant cr^2\left(\overline{v^{\theta p}}\right)^{\frac{1}{\theta}}, \qquad V_{m-1} = \left(\overline{v^\gamma_{m-1}}\right)^{\frac{1}{\gamma}} \geqslant cr^2\left(\overline{u^{\gamma q}}\right)^{\frac{1}{\gamma}}$$

and similarly

$$U_k = \left(\overline{u^\theta_k}\right)^{\frac{1}{\theta}} \geqslant cr^2\left(\overline{u^\theta}_{k+1}\right)^{\frac{1}{\theta}}, \qquad V_k = \left(\overline{v^\gamma_k}\right)^{\frac{1}{\gamma}} \geqslant cr^2\left(\overline{v^\gamma_{k+1}}\right)^{\frac{1}{\gamma}}, \quad k = 0, 1, \ldots, m-1.$$

A short calculation then yields

$$\overline{u^\theta} \leqslant cr^{-\frac{2a(m+1)\theta(s+b)}{1-ab}}.$$

Since $\Delta\overline{u^\theta} \leqslant 0$, it follows

$$N - 2 \geqslant \frac{2a(m+1)(s+b)}{1-ab}.$$

For s fixed, it is easy to see that

$$\lim_{\theta\to 0} a = \lim_{\theta\to 0} b = 1.$$

Therefore

$$\lim_{\theta\to 0} \frac{2a(m+1)(s+b)}{1-ab} = \infty,$$

contradiction. Hence no solutions exist. □

Proof of Theorem 1.4. Case $pq \leqslant 1$ follows from Lemma 3.5. Assume $pq > 1$, suppose by contradiction that (1.1) admits a non-negative and nontrivial solution (u, v). Then $u > 0$, $v > 0$ by maximum principle. If $\alpha > \frac{N-2m}{m}$, it follows from (3.5) that

$$\int_{B_R} v^p \leqslant cR^{N-2m-\frac{2m(p+1)}{pq-1}} \to 0 \quad \text{as } R \to \infty.$$

Therefore $v \equiv 0$ and so is u. If $\alpha = \frac{N-2m}{m}$, then

$$\int_{B_R} v^p \leqslant cR^{N-2m-\frac{2m(p+1)}{pq-1}} \leqslant c.$$

On the other hand, recall that for $w > 0$, $\Delta w \leqslant 0$, we have

$$w(x) \geqslant c|x|^{2-N} \quad \text{for } |x| \geqslant 1. \tag{3.6}$$

Lemma 3.1 implies

$$\overline{v_{m-1}} \geqslant cr^2\overline{u^q}, \qquad \overline{u_{m-1}} \geqslant cr^2\overline{v^p}, \qquad \overline{u_{k-1}} \geqslant cr^2\overline{u_k}, \qquad \overline{v_{k-1}} \geqslant cr^2\overline{v_k}. \tag{3.7}$$

(3.6) and (3.7) implies for $r \geqslant 1$,

$$\bar{u} \geqslant cr^{-N+2m}, \qquad \bar{v} \geqslant cr^{-N+2m}, \qquad \bar{v} \geqslant cr^{2m}\overline{u^q}. \tag{3.8}$$

If $q > 1$, we conclude from (3.8) that

$$\bar{v} \geqslant cr^{2m+(-N+2m)q} \quad \text{for } r \geqslant 1.$$

If $q < 1$, we apply Lemma 3.2 with $p_1 = qp$ and $q_1 = q$. It then follows from Lemma 3.1 that

$$\left(\overline{u^q_{m-1}}\right)^{\frac{1}{q}} \geqslant cr^2\left(\overline{v^{pq}}\right)^{\frac{1}{q}}, \qquad \left(\overline{u^q_{k-1}}\right)^{\frac{1}{q}} \geqslant cr^2\left(\overline{u^q_k}\right)^{\frac{1}{q}}.$$

From this and (3.6), we get

$$\left(\overline{u^q}\right)^{\frac{1}{q}} \geqslant cr^{-N+2m} \quad \text{for } r \geqslant 1.$$

Again we have

$$\bar{v} \geqslant cr^{2m}\overline{u^q} \geqslant cr^{2m+(-N+2m)q} \quad \text{for } r \geqslant 1.$$

Since $p > 1$ by our assumption,

$$\begin{aligned}
\int_{B_R} v^p &\geqslant \omega_N \int_1^R r^{N-1}\overline{v^p}\,dx \\
&\geqslant C\int_1^R r^{N-1}\bar{v}^p\,dr \\
&\geqslant C\int_1^R r^{N-1+2mp-(N-2m)pq}\,dr \\
&= C\ln R \to \infty.
\end{aligned}$$

Contradiction. □

Proof of Theorem 1.2. Under the additional assumption $p \geqslant 1$, $q \geqslant 1$, $(p, q) \neq (1, 1)$, we conclude from Lemma 2.5 that any positive pair of solutions (u, v) satisfies $(-\Delta)^i u > 0$, $(-\Delta)^i v > 0$, $i = 1, 2, \ldots, m-1$. The rest of the proof then follows from the same argument above. □

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