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Calculus of Variations

Minimizers near the first critical field for the nonself-dual Chern–Simons–Higgs energy

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Abstract The Chern–Simons–Higgs energy serves as a model for high temperature superconductivity. We show the existence of weak solutions to the CSH equations that are minimizers of the CSH energy. The solutions are vortexless for an applied magnetic field h_{ex} below the critical field strength, whereas vortices appear when h_{ex} exceeds the critical field strength.

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1 Introduction

Chern–Simons–Higgs (CSH) theory refers to a wide class of field theory models in $(2+1)$ dimensional Minkowski space $(\mathbb{R}^{2,1}, g)$ that contains a Chern–Simons term in the action densities [3,9,10,23]. These models have applications to the theory of high temperature superconductivity, quantum Hall effects and carry fractional charge values [3,23].

The model is described by the following CSH Lagrangian density:

$$\mathcal{L}_{\text{csh}} = D_\alpha u \overline{D^\alpha u} + \frac{\mu}{4}\epsilon^{\alpha\beta\gamma} A_\alpha \left(F_{\beta\gamma} - F^{ex}_{\beta\gamma}\right) - \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2$$

where $A = -i A_\alpha dx^\alpha$ with $A_\alpha : \mathbb{R}^{1,2} \to \mathbb{R}$ for $\alpha = 0, 1, 2$ is the gauge potential with covariant derivative $D_A = d - iA$. Here, the metric tensor $g = \text{diag}[1, -1, -1]$ is used in the

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usual way to lower and raise indices. The corresponding curvature $F_A = -\frac{1}{2} F_{\beta\gamma} dx^\beta \wedge dx^\gamma$ with $F_{\beta\gamma} = \partial_\beta A_\gamma - \partial_\gamma A_\beta$ defines the gauge field, and $u : \mathbb{R}^{1,2} \to \mathbb{C}$ is the Higgs scalar with $D_\eta u = \partial_\eta u - i A_\eta u$, $\eta = 0, 1, 2$. Furthermore, the antisymmetric Levi-Civita tensor $\epsilon^{\alpha\beta\gamma}$ is fixed by setting $\epsilon^{0,1,2} = 1$ and $\mu, \varepsilon > 0$ are the Chern–Simons coupling parameters. Here $\epsilon^{\alpha\beta\gamma} A_\alpha (F_{\beta\gamma} - F^{ex}_{\beta\gamma})$ is the Chern–Simons term with applied field tensor F^{ex}, see (1.3). The associated Euler–Lagrange equations are

$$D_\alpha D^\alpha u + \frac{1}{\varepsilon^2} u \left(|u|^2 - 1\right) \left(3\,|u|^2 - 1\right) = 0 \tag{1.1}$$

$$\frac{\mu}{4} \epsilon^{\alpha\beta\gamma} \left(F_{\beta\gamma} - F^{ex}_{\beta\gamma}\right) + \mathcal{J}^\alpha = 0 \tag{1.2}$$

where $\mathcal{J}^\alpha = (iu, D^\alpha u)$ is the matter current.

Since the $\alpha = 0$ refers to time coordinates, we replace D_0 by $\partial_\Phi = \partial_t - i\Phi$ and replace D_α by $\nabla_A = \nabla - iA$ when $\alpha \in \{1, 2\}$. Here (Φ, A) is the field potential. The curvature tensors are defined by

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & -h \\ E_2 & h & 0 \end{pmatrix}, \quad F^{ex} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -h_{ex} \\ 0 & h_{ex} & 0 \end{pmatrix}, \tag{1.3}$$

where $h = \operatorname{curl} A$, $E_\alpha = \partial_t A_\alpha - \partial_\alpha \Phi$ are the induced magnetic and electric fields and h_{ex} is the applied magnetic field. We write the current $\mathcal{J}^\alpha$ in a more classical notation by setting

$$\mathcal{J}^0 = (iu, \partial_\Phi u) = q \quad \mathcal{J}^\alpha = \left(iu, \nabla_{A_\alpha} u\right) = j^\alpha_A(u)$$

for $\alpha \in \{1, 2\}$ which are the charge and supercurrent, respectively. Therefore, the current equation reads $\frac{\mu}{2}(h - h_{ex}) + q = 0$, $-\frac{\mu}{2} E_2 + j^1_A = 0$, and $\frac{\mu}{2} E_1 + j^2_A = 0$, and in more classical notation we write the CSH equations as:

$$\partial^2_\Phi u = \nabla^2_A u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right) \left(3\,|u|^2 - 1\right) \tag{1.4}$$

$$q = -\frac{\mu}{2} \left(\operatorname{curl} A - h_{ex}\right) \tag{1.5}$$

$$j_A = \frac{\mu}{2} \left(E \times e_3\right). \tag{1.6}$$

Well-posedness questions for equations (1.4)–(1.6) were studied in [4,5].

Since $u : \mathbb{R}^2 \to \mathbb{C}$ we can easily induce the formation of topological vortices—regions where $|u| = 0$ and about which the winding number of the phase is nontrivial. Setting $u = \rho e^{i\varphi} \approx e^{i\varphi}$ over $\mathbb{R}^2$ and $\varphi = d\theta$, then $J_A \approx \frac{1}{2} \operatorname{curl} (\nabla\varphi - A) = \det \nabla u - \frac{1}{2} h$. Assuming that $E \to 0$ as $|x| \to +\infty$, then we can formally integrate (1.6) over $\mathbb{R}^2$ and get $2\pi d = \int_{\mathbb{R}^2} h dx$. Furthermore, integrating (1.5) over the plane and assuming that $h_{ex} = 0$ yields

$$d = \frac{1}{2\pi} \int_{\mathbb{R}^2} h dx = -\frac{1}{\mu\pi} \int_{\mathbb{R}^2} q dx. \tag{1.7}$$

As in Ginzburg–Landau theory, we see that the current and the magnetic field are quantized about a topological vortex; however, in CSH theory the magnetic field induces a quantized electric charge, which can have arbitrary values, depending on μ. This quantized electric charge is a fundamental feature of Chern–Simons–Higgs theory.

We look for solutions independent of time; setting $\partial_t u \equiv 0$ then (1.4)–(1.6) become

$$-\Phi^2 u = \nabla_A^2 u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right)\left(3|u|^2 - 1\right)$$
$$\Phi |u|^2 = \frac{\mu_\varepsilon}{2}\left(\operatorname{curl} A - h_{ex}\right) \quad j_A(u) = \frac{\mu}{2} \nabla \Phi \times e_3.$$

Removing the electric field potential Φ, we are left with an unusual system of coupled elliptic PDE's:

$$-\frac{\mu_\varepsilon^2}{4} \frac{|\operatorname{curl} A - h_{ex}|^2}{|u|^4} u = \nabla_A^2 u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right)\left(3|u|^2 - 1\right) \tag{1.8}$$

$$0 = -\frac{\mu_\varepsilon^2}{4} \operatorname{curl}\left(\frac{\operatorname{curl} A - h_{ex}}{|u|^2}\right) + j_A(u). \tag{1.9}$$

Equations (1.8)–(1.9) can be viewed as the Euler–Lagrange equations of the following Chern–Simons–Higgs energy

$$G_{\mathrm{csh}}(u, A; h_{ex}) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + \frac{\mu_\varepsilon^2}{4} \frac{|\operatorname{curl} A - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 dx \tag{1.10}$$

for an applied magnetic field, h_{ex}, and a bounded, simply connected domain, $\Omega \subset \mathbb{R}^2$. A discussion of the CSH theory on bounded domains can be found in [7].

If we consider a topological vortex in (1.10) with $h_{ex} = 0$ then u must vanish at least at one point. But the second term of (1.10) implies that $h = \operatorname{curl} A$ must likewise vanish at that point. On the other hand the quantization relation (1.7) implies there exists a finite mass of magnetic field about this vortex, and consequently the magnetic field concentrates in an annular region about each topological vortex. This is in contrast to Ginzburg–Landau vortices, where the magnetic field concentrates at the site of the vortex. The second term proves to greatly increase the difficulty of analyzing (1.8)–(1.9) over the Ginzburg–Landau equations, including the loss of a maximum principle.

1.1 Prior results

Most research has focussed on the self-dual case where $\varepsilon = \mu_\varepsilon$. In this case the CSH equations reduce, following Hong et al. [9] and Jackiw and Weinberg [10], to a system of first order PDE's. Solutions can be recovered by solving (after a substitution) a Liouville-type elliptic equation, similar to the Jaffe–Taubes approach to solving the self-dual Ginzburg–Landau equations [11]. Important results on self-dual solutions to the Chern–Simons–Higgs equations can be found in [3,6,7,9,10,22,23] and the references therein.

A rigorous approach to nonself-dual Chern–Simons–Higgs theory was initiated by Han and Kim in [8], where they studied existence of solutions to the CSH equations (1.8)–(1.9). Their primary result is

Theorem 1.1 (Han and Kim [8]) *Assume $u = g$ on ∂U with $|g| = 1$, and assume $h_{ex} = 0$. Then there exists a solution to* (1.8)–(1.9).

In order to establish their result, Han–Kim embed the CSH energy into an even more general Maxwell–Chern–Simons–Higgs (MCSH) energy that contains an extra neutral scalar field N. The MCSH energy has a simpler structure than the CSH energy, and the authors prove the existence of minimizers of MCSH via the direct method and estimates on the lower order terms. Then they take a limit to the CSH equations; however, the solution is not necessarily a minimizer of (1.10).

The final section of [8] studies the simplified CSH energy

$$E_{\mathrm{csh}}(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2(1-|u|^2)^2 \tag{1.11}$$

and shows that minimizers with Dirichlet boundary conditions satisfy the same convergence behavior as found by Bethuel et al. [1] for the simplified Ginzburg–Landau energy

$$E_{gl}(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{1.12}$$

Their methods are similar to those in [1] and rely heavily on the maximum principle for $|u_\varepsilon|$.

In order to study the full CSH functional (1.10) with an aim at understanding the nucleation of vortices, Kurzke and Spirn [14] placed the CSH functional in the Gamma-convergence framework. The convergence results are true for nonminimizers and even for sequences of functions that are not solutions of the corresponding equations. The Gamma-convergence results are separated into a compactness result combined with a lower bound for the energy and a construction that shows that the lower bound is essentially optimal.

Theorem 1.2 (Kurzke and Spirn [14,15]) *Let $\mu_\varepsilon \to \mu \in (0, +\infty]$ as $\varepsilon \to 0$. Assume that the external field satisfies $h_{ex} = H|\log\varepsilon|$ for some $H > 0$, and consider a sequence $\{u_\varepsilon, A_\varepsilon\}$ that satisfies the Coulomb gauge condition and*

$$G_{\mathrm{csh}}(u_\varepsilon, A_\varepsilon; h_{ex}) \le K|\log\varepsilon|^2.$$

Set $a_\varepsilon = \frac{1}{|\log\varepsilon|}A_\varepsilon$, then $\{a_\varepsilon\}$ is weakly precompact in $W^{1,p}$ for all $p < 2$, and for a subsequence such that $a_\varepsilon \rightharpoonup a$ there holds $\frac{\mathrm{curl}\, a_\varepsilon - H}{|u_\varepsilon|} \rightharpoonup \mathrm{curl}\, a - H$ in L^2.

Additionally, $v_\varepsilon = \frac{1}{|\log\varepsilon|} j(u_\varepsilon)$ converges to v weakly in all L^p with $p < 2$, $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ in L^2, and $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log\varepsilon|} \rightharpoonup w = \frac{1}{2}\,\mathrm{curl}\, v$. Taking a subsequence, the modulus $\rho_\varepsilon = |u_\varepsilon|$ satisfies $\rho_\varepsilon \to \rho$ strongly in L^p for $p < +\infty$ where ρ is either identically 0 or identically 1. If $\rho = 0$, then $\mathrm{curl}\, a = H$ and $v = 0$. Furthermore, the energy satisfies

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} G_{\mathrm{csh}}(u_\varepsilon, A_\varepsilon; h_{ex}) \ge G^{\rho}(v, a; H), \tag{1.13}$$

with

$$G^1(v, a; H) = \frac{1}{2}\left(\int_U |v-a|^2 + \frac{\mu^2}{4}|\mathrm{curl}\, a - H|^2 + \|\mathrm{curl}\, v\|_{\mathcal{M}}\right)$$

when $\mu \in (0, +\infty)$ and

$$G^1(v, a; H) = \frac{1}{2}\left(\int_U |v-a|^2 + \|\mathrm{curl}\, v\|_{\mathcal{M}}\right)$$

when $\mu = +\infty$ and

$$G^0(v, a; H) = 0. \tag{1.14}$$

Conversely, for any $a \in H^1(U;\mathbb{R}^2)$ *and* $v \in L^2(U;\mathbb{R}^2)$ *such that* $w = \frac{1}{2}\,\mathrm{curl}\, v$ *is a Radon measure, there exist a sequence* $\{u_\varepsilon\}$ *in* $H^1(U;\mathbb{C})$ *with* $|u_\varepsilon| = 1$ *on* ∂U *and a sequence* $\{A_\varepsilon\} \in H^1(U;\mathbb{C})$ *satisfying the Coulomb gauge conditions such that* $v_\varepsilon = \frac{1}{|\log\varepsilon|} j(u_\varepsilon) \rightharpoonup v$ *in* L^2, $w_\varepsilon = \frac{1}{|\log\varepsilon|} J(u_\varepsilon) \rightharpoonup w$ *in* $(C^{0,\beta})^*$, $a_\varepsilon = \frac{1}{|\log\varepsilon|} A_\varepsilon \rightharpoonup a$ *in* H^1, *and such that* (1.13) *holds with equality for* $\rho = 1$. *For* $\rho = 0$, *there exists a sequence* $(u_\varepsilon, A_\varepsilon)$ *with* $u_\varepsilon \to 0$ *and* $\frac{1}{|\log\varepsilon|}\,\mathrm{curl}\, A_\varepsilon \to H$ *such that* $G_{\mathrm{csh}}(u_\varepsilon, A_\varepsilon; h_{ex}) \to 0$.

An application of the last theorem is the following characterization of critical field. Critical field h_{c_1} is defined as the minimum applied field below which minimizers contain no vortex.

Corollary 1.3 *(See [14]) As* $\varepsilon \to 0$, *the critical field* h_{c_1} *for nontrivial local minimizers is given asymptotically by* $H_1(\mu)|\log\varepsilon|$, *where*

$$H_1(\mu) = \frac{2}{\mu^2 \max_U |z_\mu|} \tag{1.15}$$

and z_μ *is the solution of*

$$-\frac{\mu^2}{4}\Delta z_\mu + z_\mu + \ddot{a}^2/4 = 0$$

with homogeneous Dirichlet boundary conditions. When $\mu_\varepsilon \to +\infty$ *then* $H_1 = \frac{2}{\max|z|}$ *where* z *is a solution of* $\Delta z = 1$ *with homogeneous Dirichlet boundary conditions.*

Concerning the dependence on μ in (1.15), we find $\mu^2 H_1(\mu) \to 2$ as $\mu \to 0$; therefore, we formally expect that the critical field as $\mu_\varepsilon \to 0$ should be $\frac{2|\log\varepsilon|}{\mu_\varepsilon^2}$. However, when $\mu_\varepsilon \to 0$ we have:

Theorem 1.4 (Kurzke and Spirn [15]) *When* $\mu_\varepsilon \to 0$ *the* $G_{\mathrm{csh}}(u_\varepsilon, A_\varepsilon; h_{ex})$ *fails to Gamma-converge as* $\varepsilon \to 0$.

The failure of Gamma-convergence is due to the decreasing effectiveness of energy reduction via vortex nucleation. The counterexample arises from a clustering of vortices at a distance of $\frac{\mu_\varepsilon}{\sqrt{|\log\varepsilon|}}$ from each other.

1.2 Results

Although $G_{\mathrm{csh}}(u_\varepsilon, A_\varepsilon; h_{ex})$ fails to Gamma-converge when $\mu_\varepsilon \to 0$, it is natural to ask whether the added regularity of minimizers will lead to the conjectured critical field strength. This paper concerns the development of global minimizer theory for the CSH energy in a given space and we concentrate on the interesting $\mu_\varepsilon \to 0$ situation.

We define the following space

$$V = \left\{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1\left(\Omega, \mathbb{R}^2\right),\ \text{such that } |u| = 1 \text{ on } \partial\Omega\right\}.$$

Our main results are the following theorems. The first result extends the existence result of Han and Kim [8].

Theorem 1.5 *For any given* ε, μ, h_{ex}, *there exists a solution pair* (u, A) *of* (1.8), (1.9). *In particular,* (u, A) *is a minimizer of* $G_{\mathrm{csh}}(u, A; h_{ex})$ *in* V.

Our second result establishes the critical field when $\mu_\varepsilon \to 0$. The critical field calculation for $\mu_\varepsilon \to (0, +\infty]$ have already been established in [14,15].

Theorem 1.6 *Assume $\mu_\varepsilon \to 0$ satisfies $\mu_\varepsilon \gg e^{-|\log \varepsilon|^\alpha}$ for any $0 < \alpha < 1$. Then there exists a critical field $h_{c_1} = \frac{2|\log \varepsilon|}{\mu_\varepsilon^2}$ such that for $h_{ex} \le h_{c_1}$, a minimizer (u, A) in V satisfies $|u| \ge \frac{1}{4}$. For $h_{ex} > h_{c_1}$, a minimizer in V must have a vortex.*

Theorem 1.6 implies that for $\mu_\varepsilon = \frac{1}{|\log \varepsilon|^\gamma}$ for any $\gamma < +\infty$, the critical field is $\frac{2|\log \varepsilon|}{\mu_\varepsilon^2}$. When $\mu_\varepsilon \to 0$ at a faster rate, say $\mu_\varepsilon = \varepsilon^\alpha$ for some $\alpha > 0$, then our critical field proof fails.

Remark 1.7 We remark that global minimizers of $G_{\text{csh}}(u_\varepsilon, A_\varepsilon; h_{ex})$ in $H^1 \times H^1$ are trivial ($u = 0$, curl $A = h_{ex}$). Existence of nontrivial local minimizers of $G_{\text{csh}}(u_\varepsilon, A_\varepsilon; h_{ex})$ with Neumann boundary conditions seems to be a challenging problem.

Remark 1.8 There are analogous results for asymptotics of the Ginburg–Landau energy functional for asymptotically large and small domains, see [16].

1.3 Method

Due to the existence of the singular term $\frac{\mu_\varepsilon^2}{4} \frac{|\text{curl}\, A - h_{ex}|^2}{|u|^2}$ in the Chern–Simons–Higgs integrand, a standard minimization method does not yield a converging minimizing sequence in the correct space. We consider instead a penalized energy

$$G_k(u, A) = \frac{1}{2} \int_\Omega \left[|\nabla_A u|^2 + \frac{\mu_\varepsilon^2}{4} \frac{|\text{curl}\, A - h_{ex}|^2}{|u|^2 + \frac{1}{k^2}} + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \right].$$

Establishing the existence of a minimizer of G_k for k fixed is straightforward. Furthermore, the minimizer, (u_k, A_k), of G_k satisfies its associated Euler–Lagrange equations, which in turn provides better regularity estimates for (u_k, A_k), independent of k and ε. From the added regularity we are able to pass to the limit $k \to \infty$ and conclude that there exists a minimizing sequence for the original energy G_{csh} that converges to a minimizer in $H^1 \times H^1$.

In order to establish the critical field we split the energy G_{csh} into the order parameter energy E_{csh} and the magnetic field energy, similar to the splitting method of Serfaty [21] for the Ginzburg–Landau energy. Therefore, it is crucial to prove energy lower bounds on E_{csh} without assumptions on the phase. Since we do not know the number of vortices for a minimizer *a priori*, we follow an idea of Sandier and Serfaty [20] to construct disjoint balls that covers the region $|u| \le \frac{1}{2}$ for u satisfying the gradient estimate $|\nabla u| \le \frac{C}{\varepsilon}$. This method is based on a construction of Jerrard and Soner [13], Jerrard [12], and Sandier [19]. To initiate the Sandier–Serfaty framework we need a collection of balls that covers $\left\{|u| \le \frac{1}{2}\right\}$ and satisfies

$$E_{\text{csh}}(u, B_i \cap \Omega) = \int_{B_i \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \ge \frac{C r_i}{\varepsilon}.$$

In the case of the Ginzburg–Landau energy, such balls can be constructed using the lower bound estimate of the Ginzburg–Landau energy on the circle:

Lemma 1.9 (Jerrard [12]) *If $r \ge \varepsilon$ and $m = 1 \wedge \min_{\partial B_r} |u|$, then*

$$\int_{\partial B_r} \frac{1}{2} |\nabla_\tau \rho|^2 + \frac{1}{\varepsilon^2} \left(1 - \rho^2\right)^2 ds \ge \frac{(1 - m)^N}{C\varepsilon}$$

for some constant $C, N > 0$, where $\rho = |u|$.

Such an estimate fails for Chern–Simons energy $E_{\rm csh}$ due to the form of the potential term $\frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2$, which is trivial on each circle ∂B_r when $|u|$ vanishes, unlike for the Ginzburg–Landau potential. To overcome this problem, our primary observation is Lemma 2.9. The main idea is that although such a lower bound estimate may not be true for all the radii, it is true for a positive measure of radii, and this is enough to bound the Chern–Simons energy from below by $\frac{Cr_i}{\varepsilon}$. We note that our result uses crucially that (u, A) satisfies $|\nabla u| \le \frac{C}{\varepsilon}$, i.e., we need the Euler–Lagrange equations to exist. We feel our approach can handle more general potentials of the form $|u|^\alpha|1-|u|^2|^\beta$ for $\alpha, \beta > 0$. Once we have this first step initiated, the rest of the ball construction follows essentially from the arguments of Jerrard [12], Jerrard and Soner [13], and Sandier and Serfaty [20].

In order to prove the critical field, we use the energy splitting method of Bethuel and Riviere [2], Serfaty [21], and Sandier and Serfaty [20] to bound the CSH energy by:

$$\begin{aligned} G_0 &\ge G_{\rm csh}(u, A)\\ &\ge G_0 + 2\pi h_{ex}\sum_{i\in I} d_i\xi_0(a_i)\\ &\quad +\pi\sum_{i\in I}|d_i|\left(|\log\varepsilon| - O\left(\log\left|\frac{|\log\varepsilon|}{\mu_\varepsilon}\right|\right)\right), \end{aligned}$$

where $G_0 = G_{\rm csh}(1, h_{ex}\nabla^\perp\xi_0)$ is the Meissner energy and where ξ_0 solves a scaled London equation $-\frac{\mu_\varepsilon^2}{4}\Delta\xi_0 + \xi_0 = -\frac{\mu_\varepsilon^2}{4}$ in Ω with homogenous Dirichlet boundary conditions. Here I is the collection of vortices that lie away from the boundary of the domain. Since $\mu_\varepsilon \gg e^{-|\log\varepsilon|^\alpha}$ then $|\log\mu_\varepsilon| = o(|\log\varepsilon|)$. By elliptic estimates we show that $\max_\Omega|\xi_0| \sim \frac{\mu_\varepsilon^2}{4}$, and a simple comparison argument shows that the minimizer must be vortex-less when $h_{ex} < h_{c_1}$. Finally, we prove, by explicit construction, that once $h_{ex} > h_{c_1}$ there are configurations with a single vortex that have less energy than a Meissner solution.

The paper is organized as following. In Sect. 2, we state some preliminary estimates. When the Euler–Lagrange equations (1.8)–(1.9) exist, then we can prove stronger estimates and our ball construction. Section 3 is devoted to the proof of our first and second theorems.

2 Preliminaries

Let $u = \rho e^{i\varphi}: \Omega \to \mathbb{C}$ and $A: \Omega \to \mathbb{R}^2$, we consider the Chern–Simons–Higgs functional

$$G_{\rm csh}(u, A; h_{ex}) = \frac{1}{2}\int_\Omega\left[|\nabla_A u|^2 + \frac{\mu_\varepsilon^2}{4}\frac{|\text{curl}\,A - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2\right], \quad (2.1)$$

where $u \in H^1(\Omega, \mathbb{C})$, and $A \in H^1\left(\Omega, \mathbb{R}^2\right)$. $G_{\rm csh}$ is invariant under gauge transformations. More precisely, for $\phi \in H^2(\Omega, \mathbb{R})$ and

$$\begin{cases} u_\phi = e^{i\phi}u\\ A_\phi = A + d\phi. \end{cases}$$

We have

$$G_{\rm csh}(u, A; h_{ex}) = G_{\rm csh}(u_\phi, A_\phi; h_{ex}).$$

Throughout the paper, we assume Ω is a simply connected bounded domain and always choose A such that div $A = 0$ and $A\cdot\nu = 0$ on $\partial\Omega$. In particular, we can write $A = \left(-\xi_y, \xi_x\right)$ for some $\xi = 0$ on $\partial\Omega$. We define two CSH energy densities

$$g_{\rm csh}(u, A; h_{ex}) = \frac{1}{2}\left[|\nabla_A u|^2 + \frac{\mu_\varepsilon^2}{4}\frac{|{\rm curl}\, A - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2\right]$$
$$e_{\rm csh}(u) = \frac{1}{2}\left[|\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2\right],$$

and set

$$E_{\rm csh}(u) = \int_\Omega e_{\rm csh}(u)dx \quad \text{and} \quad G_{\rm csh}(u, A; h_{ex}) = \int_\Omega g_{\rm csh}(u, A; h_{ex}).$$

The associated Euler–Lagrange equations for functional (2.1) are (1.8)–(1.9).

2.1 Energy estimates

Let $|u| = \rho$. We first quote the following covering lemma on the set where $\rho < \frac{1}{2}$ from [14]. The proof of the lemma exploits the Modica–Mortola trick [17,18], used with great success by Sandier for complex Ginzburg–Landau energies [19].

Lemma 2.1 (Kurzke and Spirn [14]) *Suppose $\rho \geq \frac{3}{4}$ on $\partial\Omega$, then we have $\{x \in \Omega\colon |u| < \frac{1}{2}\} \subset \cup B_{r_j}$ with*

$$\sum r_j \leq C\varepsilon E_{\rm csh}(|u|)$$

for all $\varepsilon \leq \varepsilon_0$ small enough.

Lemma 2.1 leads to the following energy estimates.

Lemma 2.2 *Suppose $|\Omega| \leq E_{\rm csh}, G_{\rm csh} \leq M_\varepsilon$ and $\rho \geq \frac{3}{4}$ on $\partial\Omega$, then for all $2 < p < \infty$ and some small $\gamma > 0$, the following estimates hold*

$$\|\rho\|_{H^1} \leq C\sqrt{M_\varepsilon}, \tag{2.2}$$
$$\|1-\rho\|_{L^p(\Omega)} \leq C_{p,\gamma}\varepsilon^{\frac{2}{p}-\gamma}M_\varepsilon^{\frac{1}{2}+\frac{1}{p}}, \tag{2.3}$$
$$\|\rho\|_{L^p} \leq C_p. \tag{2.4}$$

Moreover, for all $1 \leq \alpha < 2$, we have bounds

$$\|j_A(u)\|_{L^\alpha} \leq C_\alpha\sqrt{M_\varepsilon}, \tag{2.5}$$
$$\|h - h_{ex}\|_{L^\alpha} \leq \frac{C_\alpha}{\mu_\varepsilon}\sqrt{M_\varepsilon}, \tag{2.6}$$

where $C_\alpha \longrightarrow \infty$ as $\alpha \longrightarrow 2$. If $\{u, A\}$ is a weak solution of (1.8)–(1.9), we have

$$\left\|\frac{h-h_{ex}}{\rho^2}\right\|_{W^{1,q}} \leq \frac{C_q}{\mu_\varepsilon^2}\sqrt{M_\varepsilon} \tag{2.7}$$

for all $1 \leq q < 2$. In particular, this implies for $A = \nabla^\perp\xi$,

$$\|\nabla\xi\|_{L^\infty} \leq \frac{C_q}{\mu_\varepsilon^2}\sqrt{M_\varepsilon} + Ch_{ex}. \tag{2.8}$$

Proof The proof of (2.2)–(2.6) can be found in [14]. We only prove (2.7) and (2.8). If u is a weak solution of (1.8)–(1.9), from $j_A(u) = \frac{\mu_\varepsilon^2}{4}\operatorname{curl}(\frac{h-h_{ex}}{\rho^2})$ we deduce that for $1 \le q < 2$,

$$\left\|\nabla\left(\frac{h-h_{ex}}{\rho^2}\right)\right\|_{L^q} = \frac{4}{\mu_\varepsilon^2}\left\|\frac{j_A(u)}{|u|}\rho\right\|_{L^q} \le \frac{C}{\mu_\varepsilon^2}\|\nabla_A u\|_{L^2}\|\rho\|_{L^{\frac{2q}{2-q}}} \le \frac{C_q}{\mu_\varepsilon^2}\sqrt{M_\varepsilon}.$$

Since $h = h_{ex}$ on $\partial\Omega$, then the Poincaré inequality implies

$$\left\|\frac{h-h_{ex}}{\rho^2}\right\|_{W^{1,q}(\Omega)} \le \frac{C_q}{\mu_\varepsilon^2}\sqrt{M_\varepsilon},$$

where C_q depends on Ω. This establishes (2.7).

Next we prove (2.8). By Sobolev embedding and (2.7), for any $1 < p < \infty$,

$$\left\|\frac{h-h_{ex}}{\rho^2}\right\|_{L^p(\Omega)} \le C_p\left\|\frac{h-h_{ex}}{\rho^2}\right\|_{W^{1,q}(\Omega)} \le \frac{C_{p,q}}{\mu_\varepsilon^2}\sqrt{M_\varepsilon}.$$

Therefore,

$$\|h-h_{ex}\|_{L^p(\Omega)} \le \left\|\frac{h-h_{ex}}{\rho^2}\right\|_{L^q(\Omega)}\|\rho^2\|_{L^r(\Omega)} \le \frac{C_{p,q}}{\mu_\varepsilon^2}\sqrt{M_\varepsilon}$$

for any $1 < p < \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Since $h = \Delta\xi$, this implies

$$\|\Delta\xi\|_{L^p(\Omega)} \le \frac{C_{p,q}}{\mu_\varepsilon^2}\sqrt{M_\varepsilon} + Ch_{ex}.$$

Since $\xi = 0$ on the boundary, then Sobolev embedding implies (2.8). □

2.2 Gradient estimate for solutions of Euler–Lagrange equation

We derive gradient estimate on solutions to Euler–Lagrange equations (1.8)–(1.9).

Lemma 2.3 *Assume* (u, A) *is a solution of* (1.8)–(1.9) *satisfying* $\frac{\partial u}{\partial n} = 0$ *on* $\partial\Omega$ *and* $G_{\text{csh}}(u, A) \le M_\varepsilon$, $h_{ex} \le \frac{\sqrt{M_\varepsilon}}{\mu_\varepsilon}$. *If* $\varepsilon\frac{M_\varepsilon}{\mu_\varepsilon} \le C$, $\varepsilon\frac{\sqrt{M_\varepsilon}}{\mu_\varepsilon} \le C$, *we have*

$$|\nabla u| \le \frac{C_0}{\varepsilon},$$

where C_0 *is a constant independent of* u, A, ε, *and* μ_ε.

Proof We follow the idea of Bethuel and Rivière [2] and Serfaty [19]. Set $\overline{x} = \frac{x}{\varepsilon}$ and

$$\begin{aligned}\overline{u} &= u(\varepsilon\overline{x}) = u(x)\\ \overline{A} &= \varepsilon A(\varepsilon\overline{x}) = \varepsilon A(x)\\ \nabla_{\overline{A}}\overline{u} &= \varepsilon\nabla_A u(\varepsilon\overline{x}) = \varepsilon\nabla_A u(x)\\ \overline{h}(\overline{x}) &= \varepsilon^2 h(\varepsilon\overline{x}) = \varepsilon^2 h(x)\\ \overline{h_{ex}} &= \varepsilon^2 h_{ex}\end{aligned}$$

then $(\overline{u}, \overline{A})$ is a critical point of

$$\overline{G}(v, B) = \frac{1}{2}\int\limits_{\overline{\Omega}} |\nabla_B v|^2 + \frac{\mu_\varepsilon^2}{4\varepsilon^2}\frac{\left|\operatorname{curl} B - \overline{h_{ex}}\right|^2}{|v|^2} + |v|^2\left(1-|v|^2\right)^2.$$

The CSH equations for this functional are

$$-\frac{\mu_\varepsilon^2}{4\varepsilon^2}\frac{\left|\overline{h}-\overline{h_{ex}}\right|^2}{|\overline{u}|^4}\overline{u} = \nabla_{\overline{A}}^2\overline{u} + \overline{u}\left(1-|\overline{u}|^2\right)\left(3|\overline{u}|^2-1\right) \tag{2.9}$$

$$\left(i\overline{u}, \nabla_{\overline{A}}\overline{u}\right) = \frac{\mu_\varepsilon^2}{4\varepsilon^2}\nabla^\perp\left(\frac{\overline{h}-\overline{h_{ex}}}{|\overline{u}|^2}\right) \tag{2.10}$$

in $\overline{\Omega}$. From rescaling we have

$$\overline{G}(\overline{u}, \overline{A}) = G_{\text{csh}}(u, A) \le M_\varepsilon;$$

therefore, by (2.6)

$$\left\|\overline{h}-\overline{h_{ex}}\right\|_{L^q(\overline{\Omega})} \le C_1\frac{\varepsilon}{\mu_\varepsilon}\sqrt{M_\varepsilon} \tag{2.11}$$

for $q < 2$, which is bounded by a constant.

Let $\overline{x}_0 \in \overline{\Omega}$.

Case 1 $\operatorname{dist}(\overline{x}_0, \partial\overline{\Omega}) > 2$. Since $\operatorname{div}\overline{A} = 0$ and $\overline{A}\cdot n = 0$ on $\partial\overline{\Omega}$, then there exists a scalar potential ξ such that

$$\Delta\xi = \overline{h} \quad \text{in } B_2(\overline{x}_0) \qquad \xi = 0 \quad \text{on } \partial B_2(\overline{x}_0).$$

By (2.11) and since $\overline{h_{ex}} \le C_2\frac{\varepsilon^2}{\mu_\varepsilon}\sqrt{M_\varepsilon}$, then

$$\left\|\overline{h}\right\|_{L^q(B_2(\overline{x}_0))} \le C_3\frac{\varepsilon}{\mu_\varepsilon}\sqrt{M_\varepsilon} = C_1\frac{\varepsilon}{\mu_\varepsilon}\sqrt{M_\varepsilon} + C_2\frac{\varepsilon}{\mu_\varepsilon}\sqrt{M_\varepsilon}.$$

By the Calderon–Zygmund inequality

$$\|\xi\|_{W^{2,q}(B_2(\overline{x}_0))} \le C_q\frac{\varepsilon}{\mu_\varepsilon}\sqrt{M_\varepsilon}$$

for all $q < 2$. Hence, by Sobolev embedding

$$\left\|\overline{A}\right\|_{L^p(B_2(x_0))} = \|\nabla\xi\|_{L^p} \le C_p\frac{\varepsilon}{\mu_\varepsilon}\sqrt{M_\varepsilon} \tag{2.12}$$

for all $p < +\infty$.

Set $\overline{u} = u_0 + u_1$ where

$$\Delta u_0 = 0 \quad \text{in } B_2(\overline{x}_0) \qquad u_0 = \overline{u} \quad \text{on } \partial B_2(\overline{x}_0) \tag{2.13}$$

and

$$\begin{aligned} -\Delta u_1 &= \overline{u}\left(1-|\overline{u}|^2\right)\left(3|\overline{u}|^2-1\right) - \overline{A}^2\overline{u} \\ &\quad -2i\overline{A}\cdot\nabla\overline{u} + \frac{\mu_\varepsilon^2}{4\varepsilon^2}\frac{\left|\overline{h}-\overline{h_{ex}}\right|^2}{|\overline{u}|^4}\overline{u} \quad \text{in } B_2(\overline{x}_0) \\ &= I + II + III + IV \\ u_1 &= 0 \quad \text{on } \partial B_2(\overline{x}_0). \end{aligned} \tag{2.14}$$

Since u_0 is harmonic, by elliptic estimates, we obtain

$$|\nabla u_0| \le C \quad \text{on } B_{\frac{3}{2}}(\overline{x}_0). \tag{2.15}$$

We now prove gradient estimate on u_1. On the right hand side of (2.14) we bound terms I–IV.

To estimate I we have for all $p < +\infty$, by (2.4)

$$\begin{aligned}&\left\|\overline{u}\left(1-|\overline{u}|^2\right)\left(3|\overline{u}|^2-1\right)\right\|_{L^p(B_2(\overline{x}_0))}\\&\quad\le C\left(\|\overline{\rho}\|_{L^p(B_2(\overline{x}_0))}+\left\|\overline{\rho}^3\right\|_{L^p(B_2(\overline{x}_0))}+\left\|\overline{\rho}^5\right\|_{L^p(B_2(\overline{x}_0))}\right)\le C_p.\end{aligned}$$

Next, to control II we use (2.12) and find

$$\begin{aligned}\left\|\overline{A}^2\overline{u}\right\|_{L^p(B_2(\overline{x}_0))}&\le\left\|\overline{A}^2\right\|_{L^q(B_2(\overline{x}_0))}\|\overline{\rho}\|_{L^r(B_2(\overline{x}_0))}\\&\le C_{p,q,r}\left\|\overline{A}\right\|^2_{W^{1,q}(B_2(\overline{x}_0))}\le C_{p,q,r}\end{aligned}$$

where $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. This holds for all $p < +\infty$. To handle the term III, we have for any $p < 2$, from (2.12)

$$\begin{aligned}&\left\|\overline{A}\cdot\nabla\overline{u}\right\|_{L^p(B_2(\overline{x}_0))}\\&\quad\le\left\|\overline{A}\cdot\nabla_{\overline{A}}\overline{u}\right\|_{L^p(B_2(\overline{x}_0))}+\left\|\overline{A}^2\overline{u}\right\|_{L^p(B_2(\overline{x}_0))}\\&\quad\le\left\|\nabla_{\overline{A}}\overline{u}\right\|_{L^2(B_2(\overline{x}_0))}\left\|\overline{A}\right\|_{L^s(B_2(\overline{x}_0))}+\left\|\overline{A}\right\|^2_{L^4(B_2(\overline{x}_0))}\|\overline{\rho}\|_{L^s(B_2(\overline{x}_0))}\le C_{p,q,r,s}\end{aligned}$$

for $\frac{1}{p}=\frac{1}{2}+\frac{1}{s}$. Finally, we estimate term IV. From (2.7) we have

$$\frac{\mu_\varepsilon^2}{4\varepsilon^2}\left\|\frac{\overline{h}-\overline{h_{ex}}}{\overline{\rho}^2}\right\|_{W^{1,q}(\overline{\Omega})}=\frac{\mu_\varepsilon^2}{4}\left\|\frac{h-h_{ex}}{\rho^2}\right\|_{W^{1,q}(\Omega)}\le C_q\sqrt{M_\varepsilon},$$

which implies from Sobolev embedding

$$\begin{aligned}\left\|\frac{\overline{h}-\overline{h_{ex}}}{\overline{\rho}^2}\right\|_{L^p(B_2(\overline{x}_0))}&\le C_{p,q}\left\|\frac{\overline{h}-\overline{h_{ex}}}{\overline{\rho}^2}\right\|_{W^{1,q}(B_2(\overline{x}_0))}\\&\le C_{p,q}\left\|\frac{\overline{h}-\overline{h_{ex}}}{\overline{\rho}^2}\right\|_{W^{1,q}(\overline{\Omega})}\le C_{p,q}\frac{\varepsilon^2}{\mu_\varepsilon^2}\sqrt{M_\varepsilon}\end{aligned}$$

for all $p < +\infty$. Therefore,

$$\begin{aligned}\left\|\frac{\mu_\varepsilon^2}{\varepsilon^2}\frac{\left|\overline{h}-\overline{h_{ex}}\right|^2}{\overline{\rho}^4}\overline{u}\right\|_{L^q(B_2(\overline{x}_0))}&\le\frac{\mu_\varepsilon^2}{\varepsilon^2}\left\|\frac{\left|\overline{h}-\overline{h_{ex}}\right|^2}{\overline{\rho}^4}\right\|_{L^r(B_2(\overline{x}_0))}\|\overline{\rho}\|_{L^s(B_2(\overline{x}_0))}\\&\le C_r\frac{\mu_\varepsilon^2}{\varepsilon^2}\left\|\frac{\overline{h}-\overline{h_{ex}}}{\overline{\rho}^2}\right\|^2_{L^{2r}(B_2(\overline{x}_0))}\\&\le C_{p,q,r}\frac{\varepsilon^2}{\mu_\varepsilon^2}M_\varepsilon\le C_{p,q,r}\end{aligned}$$

for $\frac{1}{q}=\frac{1}{s}+\frac{1}{r}$.

Combining together the estimates on the right-hand side of (2.14) yields

$$-\Delta u_1=f\quad\text{in }B_2(\overline{x}_0)\qquad u_1=0\quad\text{on }\partial B_2(\overline{x}_0)$$

with $\|f\|_{L^p(B_2(\overline{x}_0))} \leq C_p$ for $p < 2$. By the Calderon–Zygmund inequality

$$\|u_1\|_{W^{2,p}(B_2(\overline{x}_0))} \leq C_p \quad \forall p < 2,$$

where C is independent of ε, $\overline{u}$, and u_1. Hence by Sobolev embedding,

$$\|\nabla u_1\|_{L^q(B_2(\overline{x}_0))} \leq C_q \quad \forall 1 < q < \infty. \tag{2.16}$$

Combining this estimate with (2.15) yields

$$\|\nabla \overline{u}\|_{L^q(B_{3/2}(\overline{x}_0))} \leq C_q \quad \forall 1 < q < \infty. \tag{2.17}$$

Working now on $B_{3/2}(\overline{x}_0)$ instead of $B_2(\overline{x}_0)$, we see the third term III on the right hand side can be bounded by

$$\left\|\overline{A} \cdot \nabla \overline{u}\right\|_{L^p(B_{3/2}(\overline{x}_0))} \leq \left\|\overline{A}\right\|_{L^q(B_{3/2}(\overline{x}_0))} \|\nabla \overline{u}\|_{L^r(B_{3/2}(\overline{x}_0))} \leq C_{p,q,r}$$

for any $1 < p < \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. We then have

$$\|u_1\|_{W^{2,p}(B_2(\overline{x}_0))} \leq C_p \quad \forall 1 < p < \infty.$$

This yields, by Sobolev embedding $|\nabla u_1| \leq C$ on $B_{3/2}(\overline{x}_0)$, which shows, combining with (2.15), that

$$|\nabla \overline{u}| \leq C \quad \text{on } B_{3/2}(\overline{x}_0).$$

Case 2 $\overline{x}_0 \in \partial\overline{\Omega}$. We follow the idea in [19], Proposition 6.1. After a possible change of coordinate, we can assume

$$\partial\Omega \cap B_{x_0}(3\varepsilon) \subset \left\{(x_1, x_2) \in R^2 : \ x_2 = 0\right\}$$

and

$$\Omega \cap B_{x_0}(3\varepsilon) \subset \left\{(x_1, x_2) \in R^2 : \ x_2 > 0\right\}.$$

We consider the following symmetrized configuration with respect to $\partial\Omega$. We set for $(x_1, x_2) \in B_{x_0}(3\varepsilon) \cap \{x_2 \leq 0\}$,

$$\begin{aligned} u(x_1, x_2) &= u(x_1, -x_2) \\ A_1(x_1, x_2) &= 2h_{ex}x_2 + A_1(x_1, -x_2) \\ A_2(x_1, x_2) &= -A_2(x_1, -x_2). \end{aligned}$$

As

$$\begin{cases} \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \\ A \cdot n = 0 \quad \text{on } \partial\Omega \Longleftrightarrow A_2 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

∇u and A are continuous on $B_{x_0}(3\varepsilon)$. Similarly, for $x_2 \leq 0$,

$$-\Delta u\left(x_{1,} x_2\right) = -\Delta u(x - x_2),$$

$$A \cdot du(x_1, x_2) - h_{ex}x_2\frac{\partial u}{\partial x_1}\left(x_{1,} x_2\right)$$

$$= A_1\frac{\partial u}{\partial x_1}\left(x_{1,} - x_2\right) - A_2\left(-\frac{\partial u}{\partial x_2}\left(x_{1,} - x_2\right)\right) = A \cdot du(x_1, -x_2),$$

$$|A|^2 u(x_1, x_2) - h_{ex}^2 x_2^2 u(x_1, x_2) - 2h_{ex}x_2A_1u(x_1, x_2) = |A|^2 u(x_1, -x_2),$$

$$\operatorname{curl} A(x_1, x_2) - h_{ex} = h_{ex} - \operatorname{curl} A\left(x_{1,} - x_2\right).$$

Recall $\operatorname{curl} A(x_1, 0) = h_{ex}$, we deduce that $\{u, A\}$ satisfies in $B_{3\varepsilon}(x_0)$

$$-\frac{\mu_\varepsilon^2}{4}\frac{|h-h_{ex}|^2}{|u|^4}u = \nabla_A^2 u + \frac{1}{\varepsilon^2}u\left(1-|u|^2\right)\left(3|u|^2-1\right)$$
$$-x_2^-\left(2ih_{ex}\frac{\partial u}{\partial x_1} + h_{ex}^2 x_2 u + 2h_{ex}A_1 u\right)$$
$$h_{ex}\left(iu, i\begin{pmatrix} x_2^- \\ 0 \end{pmatrix}u\right) + (iu, \nabla_A u) = \frac{\mu_\varepsilon^2}{4}\operatorname{curl}\left(\frac{h-h_{ex}}{|u|^2}\right).$$

Here $x_2^- = \min(0, x_2)$. Each term in the equation is continuous acrossing the boundary. Through a similar argument as in Case 1, we deduce $|\nabla u| \leq \frac{C}{\varepsilon}$. □

2.3 Ball construction

2.3.1 Preparations

Let $\rho(x) = |u(x)|$, $m = 1 \wedge \min_{x\in\partial B_r}\rho(x)$. The following two lemmas give the lower bound estimate of the energy on the circle.

Lemma 2.4 *If $r \geq \varepsilon$, then*

$$\int_{\partial B_r} \frac{1}{2}|\nabla_\tau \rho|^2 + \frac{1}{\varepsilon^2}\rho^2\left(1-\rho^2\right)^2 ds \geq \frac{m^2(1-m)^2}{C\varepsilon}$$

for some constant $C = C(\Omega) > 0$.

Proof We follow idea of Jerrard [12]. If $\min_{x\in\partial B_r}\rho(x) > 1$, the lemma holds trivially. When $\min_{x\in\partial B_r}\rho(x) \leq 1$, let

$$\gamma = \int_{\partial B_r} \frac{1}{2}|\nabla_\tau \rho|^2$$

and $x_{\min} \in \partial B_r$ be a point where $\rho(x_{\min}) = m$. Then for any $x \in \partial B_r$

$$\begin{aligned}\rho(x) &\leq \rho(x_{\min}) + C\|\nabla_\tau \rho\|_{L^2}|x-x_{\min}|^{\frac{1}{2}} \\ &\leq m + C\gamma^{\frac{1}{2}}|x-x_{\min}|^{\frac{1}{2}} \leq \frac{1+m}{2}\end{aligned} \tag{2.18}$$

whenever $|x-x_{\min}| \leq \frac{|1-m|^2}{4C^2\gamma}$. Since $r \geq \varepsilon$ and $x_{\min} \in \partial B_r$,

$$H^1(\partial B_r \cap B_\sigma(x_{\min})) \geq C^{-1}(\sigma \wedge \varepsilon) \tag{2.19}$$

for any $\sigma > 0$. Since $\rho^2(1-\rho^2)^2 \geq C^{-1}m^2(1-m)^2$ whenever $\rho \leq \frac{1+m}{2}$, we deduce from (2.18) and (2.19) that

$$\int_{\partial B_r} \rho^2\left(1-\rho^2\right)^2 ds \geq C^{-1}m^2(1-m)^2\left(\frac{|1-m|^2}{C\gamma} \wedge \varepsilon\right).$$

It follows that

$$\int_{\partial B_r} \frac{1}{2}|\nabla_\tau \rho|^2 + \frac{1}{\varepsilon^2}\rho^2\left(1-\rho^2\right)^2 ds \geq \gamma + C^{-1}\frac{m^2(1-m)^2}{\varepsilon^2}\left(\frac{|1-m|^2}{C\gamma} \wedge \varepsilon\right).$$

The conclusion is obvious if $\varepsilon \le \frac{|1-m|^2}{C\gamma}$. If $\varepsilon > \frac{|1-m|^2}{C\gamma}$, the conclusion follows from minimization over $\gamma > 0$ and the fact that $m \in [0, 1]$. □

Lemma 2.5 *If $r \ge \varepsilon$, then*

$$\int_{\partial B_r} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 ds \ge \frac{m^2 \pi d^2}{r} + \frac{m^2 (1-m)^2}{C\varepsilon}$$

where C is given by Lemma 2.4 and $d = \deg(u, \partial B_r)$.

Proof Let $v = \frac{u}{|u|}$, then

$$\deg(v, \partial B_r) = \frac{1}{|\partial B_1|} \int_{\partial B_r} \det \nabla_\tau v ds \le \frac{1}{|\partial B_1|} \int_{\partial B_r} |\nabla_\tau v| \, ds$$
$$\le \frac{1}{|\partial B_1|} \left(\int_{\partial B_r} |\nabla_\tau v|^2 \, ds \right)^{\frac{1}{2}} |\partial B_r|^{\frac{1}{2}}.$$

It follows

$$\int_{\partial B_r} \frac{1}{2} |\nabla_\tau v|^2 \, ds \ge \frac{\pi d^2}{r}. \tag{2.20}$$

Since $|\nabla u|^2 \ge \frac{\rho^2}{2} |\nabla_\tau v|^2 + \frac{1}{2} |\nabla_\tau \rho|^2$, the conclusion follows from (2.20) and Lemma 2.4. □

Lemma 2.6 *Given $|\nabla u|_\infty \le \frac{C_0}{\varepsilon}$, there exists $\widehat{t}$ depending only on Ω such that if $\lambda_0 \varepsilon \le t < \widehat{t}$ and $\frac{1}{2} \le |u(x_0 + te^{i\theta_t})| \le \frac{3}{4}$ for some $\theta_t \in [0, 2\pi)$. Then*

$$\int_{\partial B_t(x_0) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 ds \ge \frac{C_2}{\varepsilon}$$

for some constant $C_2 = C_2(C_0, \lambda_0, \Omega)$.

Proof Let $x_t = x_0 + te^{i\theta_t}$. Since $|\nabla u|_\infty \le \frac{C_0}{\varepsilon}$ and $\frac{1}{2} \le |u(x_0 + te^{i\theta_t})| \le \frac{3}{4}$,

$$\left| u\left(x_0 + te^{i\theta}\right) - u\left(x_0 + te^{i\theta_t}\right) \right| \le \frac{C_0}{\varepsilon} t \, |\theta - \theta_t| .$$

It follows that

$$\frac{3}{8} \le \left| u\left(x_0 + te^{i\theta}\right) \right| \le \frac{7}{8}$$

whenever $|\theta - \theta_t| \le \frac{1}{8C_0 t} \varepsilon$. Since $\partial\Omega$ is smooth, we can find a constant $\alpha > 0$ and $\widehat{t}$ such that for $t < \widehat{t}$ and any $x_0 \in \Omega$,

$$H^1 \left(\partial B_t(x_0) \cap \Omega \cap B_{\frac{1}{8C_0}\varepsilon}(x_t) \right) \ge \alpha (t \wedge \varepsilon),$$

where α is a constant depending only on Ω. (A similar type of argument is given by Jerrard [12, formula (2.8), p. 728]). Therefore, if $t \geq \lambda_0 \varepsilon$,

$$\int\limits_{\Omega \cap \partial B_t} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 ds \geq \int\limits_{\Omega \cap \partial B_t} \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 ds$$
$$\geq \left(\frac{3}{8}\right)^2 \left(\frac{1}{8}\right)^2 \frac{1}{\varepsilon^2} \left|\partial B_t \cap \Omega \cap B_{\frac{1}{8C_0}\varepsilon}(x_t)\right|$$
$$\geq \frac{C_2}{\varepsilon}.$$

□

Remark 2.7 Given $|\nabla u_\varepsilon|_\infty \leq \frac{C_0}{\varepsilon}$. If $|u_\varepsilon(x_0 + te^{i\theta_t})| = \frac{3}{4}$ for some $\theta_t \in [0, 2\pi)$ and $t \ll \varepsilon$, then $|u(z)| \geq \frac{1}{2}$ for $z \in B_t(x_0)$. In fact, if $|u(z_0)| < \frac{1}{2}$ for some $z_0 \in B_t(x_0)$, then

$$\begin{aligned}\frac{1}{4} &\leq \left|u_\varepsilon\left(x_0 + te^{i\theta_t}\right)\right| - |u(z_0)| \\ &\leq \left|u_\varepsilon\left(x_0 + te^{i\theta_t}\right) - u(z_0)\right| \\ &\leq \frac{C_0}{\varepsilon}\left|x_0 + te^{i\theta_t} - z_0\right| \leq \frac{2C_0}{\varepsilon} t.\end{aligned}$$

which contradicts the assumption on t.

Lemma 2.8 *If $|u(x_0)| = \frac{3}{4}$ and $|\nabla u|_\infty \leq \frac{C_0}{\varepsilon}$, then there exist constants $\mu_0, \nu_0 > 0$ such that*

$$\int\limits_{B_{\mu_0 \varepsilon}(x_0) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \geq \nu_0.$$

Here $\mu_0 = \mu_0(C_0)$, $\nu_0 = \nu_0(C_0, \Omega)$.

Proof Since $|u(x) - u(x_0)| \leq |\nabla u|_\infty |x - x_0| \leq \frac{C_0}{\varepsilon} |x - x_0|$, it follows that $\frac{3}{4} - \frac{C_0}{\varepsilon}\rho \leq |u(x)| \leq \frac{3}{4} + \frac{C_0}{\varepsilon}\rho$ for $x \in B_\rho(x_0) \cap \Omega$. Let $\rho = \frac{1}{8C_0}\varepsilon$, then

$$\frac{1}{8} \leq |u(x)| \leq \frac{7}{8}, \quad \frac{1}{8} \leq 1 - |u(x)| \leq \frac{7}{8}$$

for $x \in B_{\frac{1}{8C_0}\varepsilon}(x_0) \cap \Omega$. It follows that

$$\int\limits_{B_{\frac{1}{8C_0}\varepsilon}(x_0) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \geq \left(\frac{1}{8}\right)^4 \frac{1}{\varepsilon^2} \left|B_{\frac{1}{8C_0}\varepsilon}(x_0) \cap \Omega\right|. \tag{2.21}$$

Since Ω is a smooth domain, we can find $\alpha = \alpha(\Omega) > 0$ such that

$$|B_r(x_0) \cap \Omega| \geq \alpha r^2 \tag{2.22}$$

for $x \in \Omega$ and $\forall 0 < r \leq 1$. We deduce from (2.21) and (2.22) that

$$\int\limits_{B_{\frac{1}{8C_0}\varepsilon}(x_0) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \geq \left(\frac{1}{8}\right)^4 \frac{1}{\varepsilon^2} \alpha \left(\frac{1}{8C_0}\varepsilon\right)^2 = \nu_0.$$

□

2.3.2 Ball construction

Throughout this subsection, we shall assume $|u| \geq \frac{3}{4}$ on $\partial\Omega$ and $G_{\text{csh}} \leq M_\varepsilon = C\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. Central to the proof of our critical field theorem is the following lemma that initializes a set of vortex balls with the correct size and correct energy.

Lemma 2.9 *Let $S_1, \ldots, S_k$ be connected components of $\{|u_\varepsilon(x)| < \frac{3}{4}\}$ that intersect $\{|u_\varepsilon(x)| < \frac{1}{2}\}$. There exist constants $\varepsilon_0, \delta > 0$ and balls B_j centered at x_j and of radii r_j, $j = \{1, \ldots, l\}$ such that when $\varepsilon < \varepsilon_0$,*

$$\cup_{i=1}^{k} S_i \subset \cup_{j=1}^{l} B_j, \tag{2.23}$$

and for any $j \in \{1, \ldots, l\}$

$$L_j = \left\{ t \in \left(0, r_j\right) : \partial B_t\left(x_j\right) \cap \left\{ |u_\varepsilon(x)| \geq \frac{1}{2} \right\} \neq \emptyset \right\}$$

satisfies

$$\left|L_j\right| \geq \delta r_j. \tag{2.24}$$

Moreover, there exists a constant $C_2 = C_2(\delta, \Omega)$ such that for $i = \{1, \ldots, l\}$,

$$\int_{B_i(x_i)\cap\Omega} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 ds \geq \frac{C_2}{\varepsilon} r_i. \tag{2.25}$$

Proof To prove this claim, we pick $x_i \in \partial S_i$ and define

$$r_i = \sup\left\{ r > 0 : \partial B_t(x_i) \cap \left\{ |u(x)| < \frac{3}{4} \right\} \neq \emptyset \quad \forall 0 < t < r \right\},$$
$$L_i = \left\{ t \in (0, r_i) : \partial B_t(x_i) \cap \left\{ |u(x)| \geq \frac{1}{2} \right\} \neq \emptyset \right\}.$$

We discuss two cases. The first case handles large balls, and the desired lower bound follows quickly. The second case handles small balls. The small-ball case is rather difficult, and we need to increase the size of the ball by iteration until the lower bound holds or we attain the size of the ball in Case 1.

Case 1 $r_i > 2C\varepsilon\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. Here C is a constant depending only on Ω and the estimate constant C_5 given by the covering argument for $\{|u(x)| < \frac{1}{2}\}$ in Lemma 2.1. Recall that there exists constants $\alpha = \alpha(\Omega)$ and $\widehat{r}$ such that for all $0 < r < \widehat{r}$ and for all $x \in \Omega$,

$$|B_r(x) \cap \Omega| \geq \alpha r^2. \tag{2.26}$$

On the other hand,

$$\begin{aligned} \left|B_{r_i}(x_i) \cap \Omega\right| &= \int_{B_{r_i}(x_i)\cap\Omega} 1dx = \int_0^{r_i} |\partial B_t(x_i) \cap \Omega|\, dt \\ &= \int_{L_i} |\partial B_t(x_i) \cap \Omega|\, dt + \int_{[0,r_i]\setminus L_i} |\partial B_t(x_i) \cap \Omega|\, dt \\ &\leq 2\pi r_i |L_i| + r_i C_5 \varepsilon \frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}. \end{aligned} \tag{2.27}$$

Where in the last inequality, we used Lemma 2.1 and the assumption $G_{\mathrm{csh}} \leq M_\varepsilon = C\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. It follows from (2.26) and (2.27) that if

$$r_i > \frac{2C_5}{\alpha}\varepsilon\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$$

then

$$|L_i| \geq \frac{\alpha}{4\pi} r_i, \tag{2.28}$$

i.e., for $C = \frac{C_5}{\alpha}$ and $r_i > 2C\varepsilon\frac{\log^2\varepsilon}{\mu_\varepsilon^2}$, then (2.24) holds with $\delta_1 = \frac{\alpha}{4\pi}$.

To prove the energy bound (2.25), we note for any $t \in L_i$, we have

$$\partial B_t(x_i) \cap \left\{|u| \geq \frac{1}{2}\right\} \neq \emptyset, \tag{2.29}$$

and

$$\partial B_t(x_i) \cap \left\{|u| < \frac{3}{4}\right\} \neq \emptyset. \tag{2.30}$$

By continuity of u, this implies the existence of a point $y \in \partial B_t(x_i)$ such that $\frac{1}{2} \leq |u| \leq \frac{3}{4}$. In fact, if $\partial B_t(x_i) \cap \{\frac{1}{2} \leq |u| \leq \frac{3}{4}\} = \emptyset$, from (2.29) we must have

$$\partial B_t(x_i) \cap \left\{|u| > \frac{3}{4}\right\} \neq \emptyset.$$

This and (2.30) implies there exists a point $y \in \partial B_t(x_i)$ with $|u(y)| = \frac{3}{4}$, contradiction. Since $\frac{1}{2} \leq |u(y)| \leq \frac{3}{4}$, it follows from Lemma 2.6 that

$$\int_{\partial B_t(x_i)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 ds \geq \frac{C_1}{\varepsilon}.$$

From this and (2.28) we conclude

$$\begin{aligned}
&\int_{B_{r_i}(x_i)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 dx \\
&\quad \geq \int_{L_i}\int_{\partial B_t(x_i)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 ds dt \\
&\quad \geq \frac{\alpha}{4\pi} r_i \cdot \frac{C_1}{\varepsilon} = \frac{\alpha C_1}{4\pi}\cdot\frac{r_i}{\varepsilon}.
\end{aligned}$$

Case 2 $r_i \leq 2C\varepsilon\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. Let $\delta_0 = \min(\delta_1, \frac{\beta}{3})$, where $\delta_1 = \frac{\alpha}{4\pi}$ is from Case 1 and $\beta \in (0,1)$ is a constant determined later. We make our choice in the following way.

Case 2.A (2.24) holds with $\delta = \delta_0$.

In this case we keep this choice of x_i and $B_{r_i}(x_i)$. The energy bound in this case can be proved in the same way as Case 1.

Case 2.B (2.24) fails for $\delta = \delta_0$.

In this case, we replace $B_{r_i}(x_i)$ by another ball through a replacement procedure.

First we introduce a ball of twice the size of $B_{r_i}(x_i)$. By the definition of r_i, $|u(x)| \geq \frac{3}{4}$ on $\partial B_{r_i}(x_i)$ and there must exist a point $P \in \partial(B_{r_i}(x_i) \cap \Omega)$ such that $|u(P)| = \frac{3}{4}$. Otherwise if $|u(x)| > \frac{3}{4}$ on $\partial(B_{r_i}(x_i) \cap \Omega)$, then there exists a small $\eta > 0$ such that $|u(x)| > \frac{3}{4}$ on $\partial(B_t(x_i) \cap \Omega)$ for $r_i - \eta < t < r_i$. Contradiction to the definition of r_i. Pick this P and consider $B_{2r_i}(P)$. Let $\widetilde{L_i} = \{t \in (0, 2r_i) : \partial B_t(P) \cap \{|u(x)| \geq \frac{1}{2}\} \neq \emptyset\}$. We claim that there exists a constant $\beta > 0$ such that

$$\left|\widetilde{L_i}\right| \geq \beta \cdot 2r_i. \tag{2.31}$$

and there exists a constant $C > 0$ such that

$$\int\limits_{B_{2r_i}(P)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 dx \geq \frac{C \cdot 2r_i}{\varepsilon}. \tag{2.32}$$

Proof of (2.31) There are two cases.

Case a $B_{r_i}(x_i) \subset \Omega$. For such balls, we have $\forall t \in (0, 2r_i)$,

$$\partial B_t(P) \cap \partial B_{r_i}(x_i) \neq \emptyset, \tag{2.33}$$

therefore (2.31) holds trivially with $\beta = 1$.

Case b $B_{r_i}(x_i) \cap \partial\Omega \neq \emptyset$. For those balls intersecting the boundary, we have two possibilities. If we can find $P \in \partial B_{r_i}(x_i)$, since there exists constant $\kappa > 0$ that

$$\left|\partial B_{r_i}(x_i) \cap \Omega\right| \geq \kappa r_i$$

when $\varepsilon \leq \varepsilon_0$. This implies (2.33) holds for $\forall t \in (0, cr_i)$, where c is a constant depending on κ. It follows (2.31) holds with a constant β depending on κ. If no such P exists, we must have $|u| > \frac{3}{4}$ on $\partial B_{r_i}(x_i)$ and there exists $Q \in \partial\Omega \cap B_{r_i}(x_i)$ such that $|u(Q)| = \frac{3}{4}$. In this case, we choose our P such that

$$\operatorname{dist}\left(P, \partial B_{r_i}(x_i)\right) \leq \operatorname{dist}\left(Q, \partial B_{r_i}(x_i)\right)$$

for all $Q \in \partial\Omega$ satisfying $|u(Q)| = \frac{3}{4}$. For this choice of P,

$$\partial B_t(P) \cap \partial B_{r_i}(x_i) \neq \emptyset, \quad \forall t \in \left(0, r_i + \operatorname{dist}\left(P, \partial B_{r_i}(x_i)\right)\right).$$

Therefore, (2.31) holds with $\beta = \frac{1}{2}$.

Proof of (2.32) We note since (2.24) fails for $\delta = \delta_0$, for all $t \in [0, r_i] \backslash L_i$, since

$$\partial B_t(x_i) \subset \left\{|u| < \frac{1}{2}\right\},$$

we deduce

$$\partial B_{r_i - t}(P) \cap \left\{|u| < \frac{1}{2}\right\} \neq \emptyset,$$

and

$$|[0, r_i] \backslash L_i| \geq (1-\delta)\, r_i. \tag{2.34}$$

Case a $\beta \geq \frac{1}{2}$. Since for all $t \in (0, r_i)$

$$\partial B_{r_i - t}(P) \cap \left\{|u| \geq \frac{1}{2}\right\} \neq \emptyset.$$

This implies $r_i - t \in \widetilde{L}_i$, for all $t \in [0, r_i] \setminus L_i$. A similar argument as in Case 1 yields

$$\int_{\partial B_{r_i - t}(P) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 ds \geq \frac{C_1}{\varepsilon}. \tag{2.35}$$

This and (2.34) implies

$$\begin{aligned}
&\int_{B_{2r_i}(P) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \\
&\quad = \int_0^{2r_i} \int_{\partial B_t(P) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \\
&\quad \geq \int_{t \in [0, r_i] \setminus L_i} \int_{\partial B_{r_i - t}(P) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \\
&\quad \geq \frac{C_1}{\varepsilon} (1 - \delta) r_i = \frac{C_1}{2\varepsilon} (1 - \delta) \cdot 2r_i,
\end{aligned} \tag{2.36}$$

Case b $\beta < \frac{1}{2}$. In this case we have $\beta > 2\delta$ and $|L_i| \leq \delta r_i$, this yields

$$|[0, \beta r_i] \setminus L_i| \geq \frac{\beta}{2} r_i.$$

This and (2.35) implies

$$\begin{aligned}
&\int_{B_{2r_i}(P) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \\
&\quad \geq \int_0^{\beta r_i} \int_{\partial B_t(P) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \\
&\quad \geq \int_{\{t \in [0, \beta r_i]: r_i - t \in [0, r_i] \setminus L_i\}} \int_{\partial B_t(P) \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 \\
&\quad \geq \frac{\beta}{2} r_i \frac{C_1}{\varepsilon} = \frac{C_1 \beta}{4\varepsilon} \cdot 2r_i
\end{aligned} \tag{2.37}$$

(2.36) and (2.37) yields (2.32) with $C = \min\left(\frac{C_1 \beta}{4}, \frac{C_1}{2}(1 - \delta)\right)$.

We now start our replacement procedure.

Case 2.B.I $\partial B_{2r_i}(P) \cap \left\{|u(x)| < \frac{3}{4}\right\} = \emptyset$.

In this case we replace x_i by P and $B_{r_i}(x_i)$ by $B_{2r_i}(P)$ and stop. From (2.31), (2.24) holds for the new choices with $\delta = \beta$ and (2.32) gives (2.25) for this new choice.

Case 2.B.II $\partial B_{2r_i}(P) \cap \{|u(x)| < \frac{3}{4}\} \neq \emptyset$.

In this case, we consider $B_{\widetilde{r}}(P)$ where $\widetilde{r}$ is defined by

$$\widetilde{r} = \sup_r \left\{ r > 2r_i : \partial B_t(P) \cap \left\{ |u(x)| < \frac{3}{4} \right\} \neq \emptyset \text{ for all } 2r_i < t < r \right\}.$$

Case 2.B.II.a $\widetilde{r} < 4r_i$.

Consider

$$\widetilde{L} = \left\{ t \in (0, \widetilde{r}) : \partial B_t(P) \cap \left\{ |u(x)| \geq \frac{1}{2} \right\} \neq \emptyset \right\},$$

we have

$$\begin{aligned} \left|\widetilde{L}\right| &\geq \left| \left\{ t \in (0, 2r_i) : \partial B_t(P) \cap \left\{ |u(x)| \geq \frac{1}{2} \right\} \neq \emptyset \right\} \right| \\ &\geq \beta 2r_i \geq \frac{\beta}{2}\widetilde{r}. \end{aligned}$$

In this case, we replace x_i by P and $B_{r_i}(x_i)$ by $B_{\widetilde{r}}(P)$ and stop. Clearly, (2.24) holds with $\delta = \frac{\beta}{2}$. Moreover, from (2.32),

$$\begin{aligned} &\int\limits_{B_{\widetilde{r}}(P)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 dx \\ &\quad\geq \int\limits_{B_{2r_i}(P)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 dx \\ &\quad\geq \frac{C \cdot 2r_i}{\varepsilon} \geq \frac{C}{2\varepsilon}\widetilde{r}, \end{aligned}$$

(2.25) in this case follows with $C_2 = \frac{C}{2}$.

Case 2.B.II.b $\widetilde{r} > 4r_i$ and satisfy (2.24) with $\delta = \delta_0$, we replace x_i by P and $B_{r_i}(x_i)$ by $B_{\widetilde{r}}(P)$ and stop. We prove the energy bound (2.25) in this case. Consider

$$\widetilde{\widetilde{L}} = \left\{ t \in (2r_i, \widetilde{r}) : \partial B_t(P) \cap \left\{ |u| \geq \frac{1}{2} \right\} \neq \emptyset \right\}.$$

If $\left|\widetilde{\widetilde{L}}\right| < \frac{\delta}{2}(\widetilde{r} - 2r_i)$, since $\left|\widetilde{L}\right| \geq \delta\widetilde{r}$,

$$\frac{\delta}{2}(\widetilde{r} - 2r_i) + 2r_i \geq \left|\widetilde{L}\right| \geq \delta\widetilde{r},$$

i.e.,

$$2r_i \geq \frac{\delta}{2 - \delta}\widetilde{r}.$$

It then follows from (2.32) that

$$\begin{aligned} &\int\limits_{B_{\widetilde{r}}(P)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 dx \\ &\quad\geq \int\limits_{B_{2r_i}(P)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 dx \\ &\quad\geq \frac{C \cdot 2r_i}{\varepsilon} \geq \frac{C}{\varepsilon} \cdot \frac{\delta}{2 - \delta}\widetilde{r}. \end{aligned}$$

On the other hand if $|\widetilde{\widetilde{L}}| \geq \frac{\delta}{2}(\widetilde{r} - 2r_i)$, then recall $\forall t \in (2r_i, \widetilde{r})$,

$$\partial B_t(P) \cap \left\{ |u| < \frac{3}{4} \right\} \neq \emptyset.$$

For any $t \in \widetilde{\widetilde{L}}$, a similar argument as in Case 1 implies

$$\int\limits_{\partial B_t(x_i)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 ds \geq \frac{C_1}{\varepsilon}.$$

Therefore,

$$\begin{aligned}&\int\limits_{B_{\widetilde{r}}(P)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 dx\\ &\quad\geq \int\limits_{2r_i}^{\widetilde{r}} \int\limits_{\partial B_t(x_i)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 ds\\ &\quad\geq \int\limits_{\widetilde{\widetilde{L}}} \int\limits_{\partial B_t(x_i)\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 ds\\ &\quad\geq \frac{\delta}{2}\left(\widetilde{r}-2r_i\right)\cdot\frac{C_1}{\varepsilon} \geq \frac{C_1\delta}{4}\frac{\widetilde{r}}{\varepsilon}.\end{aligned}$$

Case 2.B.II.c $\widetilde{r} > 4r_i$ and (2.24) fails with $\delta = \delta_0$, we replace x_i by P and $B_{r_i}(x_i)$ by $B_{\widetilde{r}}(P)$ and start the process again. Keep repeating the process over and over again as needed until either we stop somewhere in the middle or we reach the radius $r_i > 2C\varepsilon\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. We then apply Case 1. In any case, we have (2.24) holds for $\delta = \delta_0$ and (2.25) for each of the chosen balls.

Finally, (2.23) follows directly from our construction. □

Next we follow the framework of Sandier and Serfaty [20] to finish the ball construction. In the following we let

$$E_{\text{csh}}(u, D) = \int\limits_D e_{\text{csh}}(u)dx,$$

where $D \subseteq \Omega$.

Lemma 2.10 *$u\colon \Omega \longrightarrow \mathbb{C}$ satisfying $|\nabla u| \leq \frac{C_0}{\varepsilon}$. There exist a constant $\lambda_0 > 0$ and disjoint balls $B_1, \ldots, B_l$ of radii r_i such that*

1. *for all $1 \leq i \leq l$, $r_i \geq \lambda_0\varepsilon$.*
2. *$\{|u(x)| < \frac{1}{2}\} \subset \cup_i B_i$, and $\forall 1 \leq i \leq l$, $\{|u(x)| < \frac{1}{2}\} \cap B_i \neq \emptyset$.*
3. *$\forall 1 \leq i \leq l$,*

$$E_{\text{csh}}(u, B_i\cap\Omega) = \int\limits_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 \geq \frac{C_3 r_i}{\varepsilon}.$$

Proof First, we include $\{|u(x)| < \frac{1}{2}\}$ in balls. We use the balls obtained in Lemma 2.9 and from (2.25), we have

$$\int\limits_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 dx \geq \frac{C_3 r_i}{\varepsilon}. \tag{2.38}$$

We now repeat Steps 2–5 of proof of Lemma 3.1 in Sandier and Serfaty [20] to finish the proof. □

Lemma 2.11 *For all $r > s > \lambda_0\varepsilon$, if B_r and B_s are two concentric balls of respective radii r and s and if $u\colon B_r\backslash\overline{B_s} \longrightarrow \mathbb{C}$ is such that $|u| > \frac{1}{2}$, $d = \deg(u, \partial B_r)$. Then*

$$\int\limits_{B_r\backslash\overline{B_s}\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 dx \geq |d|\left(\Lambda_\varepsilon\left(\frac{r}{|d|}\right) - \Lambda_\varepsilon\left(\frac{s}{|d|}\right)\right)$$

where Λ_ε is a function satisfies the following properties:

1. $\frac{\Lambda_\varepsilon(s)}{s}$ *is decreasing on $\mathbb{R}_+$.*
2. $\sup_{s\in\mathbb{R}_+}\frac{\Lambda_\varepsilon(s)}{s} \leq \frac{C}{\varepsilon}$.
3. *There exist ε_0, $t_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $\lambda_0\varepsilon < t < t_0$, then*

$$\left|\Lambda_\varepsilon(t) - \pi\log\frac{t}{\varepsilon}\right| \leq C.$$

Proof By Lemma 2.5, for $\lambda_0\varepsilon < s < t < r$,

$$\begin{aligned}\int\limits_{\partial B_t} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 ds &\geq \frac{m^2\pi d^2}{t} + \frac{m^2(1-m)^2}{C\varepsilon}\\ &\geq \frac{m^2\pi d^2}{t} + \frac{\left(\frac{1}{2}\right)^2(1-m)^2}{C\varepsilon}\\ &= \frac{m^2\pi d^2}{t} + \frac{C_4(1-m)^2}{\varepsilon}.\end{aligned}$$

Minimize the right hand side with respect to $m \in [0, 1]$, we yield

$$\int\limits_{\partial B_t} |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 ds \geq f_\varepsilon\left(\frac{t}{|d|}\right)$$

with

$$f_\varepsilon(s) = \frac{ab}{a+b}, \qquad a = \frac{\pi}{s}, \quad b = \frac{C_4}{\varepsilon}.$$

We then repeat the proof of Lemma 3.2 in Sandier and Serfaty [20] to finish the proof. □

We quote the following proposition from Sandier–Serfaty.

Proposition 2.12 [20, Proposition 3.1] *Let $u\colon\Omega \longrightarrow \mathbb{C}$ be such that $|\nabla u|_\infty \leq \frac{C_0}{\varepsilon}$, and $\{B_i\}$ be a family of balls of radii r_i satisfying result of Lemma 2.10. Let*

$$d_i = \begin{cases}\deg(u, \partial B_i) & \text{if } \overline{B_i} \subset \Omega\\ 0 & \text{otherwise.}\end{cases}$$

Let also $s_0 = \min_{i,d_i\neq 0}\frac{r_i}{|d_i|}$. Then for every $s \geq s_0$, there exists a family $\mathcal{B}(s)$ of disjoint balls $B_1(s), \ldots, B_{k(s)}(s)$ of radii $r_i(s)$ such that

1. *The family of balls is monotone, i.e., if $s < t$, then*

$$\cup_i B_i(s) \subset \cup_i B_i(t).$$

2. *For every i, $E_{\text{csh}}(u, B_i(s)) \geq r_i(s)\frac{\Lambda_\varepsilon(s)}{s}$ with $\Lambda_\varepsilon(s)$ defined by Lemma 2.11.*
3. *If $\overline{B_i(s)} \subset \Omega$, $d_i(s) = \deg(u, \partial B_i(s))$, then $r_i \geq s|d_i(s)|$.*

We now have the final balls from the following proposition.

Proposition 2.13 *$u : \Omega \longrightarrow \mathbb{C}$ be such that $|\nabla u|_\infty \le \frac{C_0}{\varepsilon}$, and $E_{\text{csh}}(u) \le C\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. Then for any $\alpha > 0$, $\exists$ disjoint balls $\{B_i\}_{i\in I}$ such that for sufficiently small ε*

1. $\{|u(x)| < \frac{1}{2}\} \subset \cup_i B_i$.
2. $\text{card } I \le C\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$.
3. $r_i \le C(\frac{|\log\varepsilon|}{\mu_\varepsilon})^{-\alpha}\frac{1}{\mu_\varepsilon}$.
4. *If $\overline{B_i} \subset \Omega$ and $d_i = \deg(u, \partial B_i)$, then*

$$E_{\text{csh}}(u, B_i) \ge \pi |d_i| \left(|\log\varepsilon| - O\left(\log\left|\frac{|\log\varepsilon|}{\mu_\varepsilon}\right|\right)\right). \tag{2.39}$$

Proof We start with the balls given by Lemma 2.10 then get larger balls by Proposition 2.12. We check if s_0 is small enough to be able to apply Proposition 2.12 for s large enough. Indeed, $s_0 = \min_{d_i\neq 0}\frac{r_i}{|d_i|}$, but from Lemma 2.10,

$$Cr_i \le \varepsilon E_{\text{csh}}(u, B_i \cap \Omega) \le C\varepsilon\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2},$$

so that $s_0 \le C\varepsilon\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. We can then apply Proposition 2.12 for all $s \ge C\varepsilon\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$. In particular, we choose

$$s_1 = \left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^{-\alpha-1}.$$

Proposition 2.12 yields final balls $\mathcal{B}(s_1)$ such that for all i

$$\text{if } \overline{B_i} \subset \Omega, \quad E_{\text{csh}}(u, B_i(s)) \ge \frac{\Lambda_\varepsilon(s_1)}{s_1} r_i(s_1), \tag{2.40}$$

with $r_i(s_1) \ge s_1 |d_i(s_1)|$. Therefore,

$$E_{\text{csh}}(u, B_i(s_1)) \ge \Lambda_\varepsilon(s_1)|d_i(s_1)|,$$

and from Lemma 2.11,

$$E_{\text{csh}}(u, B_i(s_1)) \ge |d_i(s_1)|\left(\pi\log\frac{s_1}{\varepsilon} - C\right) \ge |d_i(s_1)|\left(\pi|\log\varepsilon| - O\left(\log\left|\frac{|\log\varepsilon|}{\mu_\varepsilon}\right|\right)\right).$$

This proves the lower bound on E_{csh}.

To prove the third assertion, we get from (2.40), that

$$\frac{\Lambda_\varepsilon(s_1)}{s_1} r_i(s_1) \le E_{\text{csh}}(u, B_i(s)) \le C\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}. \tag{2.41}$$

Since $s_1 = \left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^{-\alpha-1}$ and $\Lambda_\varepsilon(s_1) \ge \pi|\log\varepsilon| - O\left(\log\left|\frac{|\log\varepsilon|}{\mu_\varepsilon}\right|\right)$, it follows from (2.41) that

$$r_i(s_1) \le \frac{C\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}\left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^{-\alpha-1}}{|\log\varepsilon|} \le C\left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^{-\alpha}\frac{1}{\mu_\varepsilon}.$$

To prove the second assertion, we recall that from Lemma 2.10 each ball satisfies

$$E_{\text{csh}}(u, B_i) \ge C\frac{r_i}{\varepsilon}$$

with $r_i \geq \lambda_0\varepsilon$, hence carries an energy that is bounded from below by a constant independent of ε. As $E_{\text{csh}} \leq C\frac{|\log\varepsilon|^2}{\mu_\varepsilon^2}$ and the procedure of Proposition 2.12 does not increase the number of balls, the bound on the number of balls follows. □

3 Proof of Theorems 1.5 and 1.6

3.1 Existence of minimizer

In this section, we prove Theorem 1.5. We consider the minimization problem of the functional (2.1) in the space

$$V = \left\{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1\left(\Omega, \mathbb{R}^2\right), \text{ such that } |u| = 1 \text{ on } \partial\Omega\right\}.$$

We have

Lemma 3.1 *For all $\varepsilon, \mu_\varepsilon > 0$, h_{ex} given, there exists a solution $(u_\varepsilon, A_\varepsilon)$ of (1.8)–(1.9) satisfying $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ and $(u_\varepsilon, A_\varepsilon)$ is a minimizer of $G_{\text{csh}}(u, A; h_{ex})$ in V.*

Proof Let $W = \cap_{2>p>1} W^{1,p}$. We prove the existence in two steps. For simplicity of notation, we write $G_{\text{csh}}(u, A; h_{ex})$ as $G(u, A)$.
Step 1: Let

$$G_k(u, A) = \frac{1}{2}\int_\Omega \left[|\nabla_A u|^2 + \frac{\mu^2}{4}\frac{|\text{curl}\, A - h_{ex}|^2}{|u|^2 + \frac{1}{k^2}} + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2\right].$$

We consider the penalized minimization problem $\inf_V G_k(u, A)$. We claim there is a minimzer $(v_{k,} B_k)$ for $G_k(u, A)$ in V. Let (v_n, A_n) is a minimizing sequence for $G_k(u, A)$ in $H^1 \times W$ and $|v_n| = 1$ on $\partial\Omega$. We choose A_n such that $\text{div}\, A_n = 0$ and $A_n \cdot \nu = 0$ on $\partial\Omega$ and write $A_n = (-\xi_{ny}, \xi_{nx})$. By Lemma 2.2 , we have for all $1 \leq p < 2$,

$$\|h_n - h_{ex}\|_{L^p} \leq C, \quad \left\|\nabla_{A_n} u_n\right\|_{L^2} \leq C \tag{3.1}$$
$$\text{and} \quad \|\rho_n\|_{H^1} \leq C. \tag{3.2}$$

By Sobolev embedding, (3.2) implies

$$\|\rho_n\|_{L^q} \leq C \quad \forall 1 < q < \infty. \tag{3.3}$$

Since $\Delta\xi_n = \text{curl}\, A_n = h_n$, by $W^{1,p}$ estimates for elliptic equations, for all $1 < p < 2$,

$$\|A_n\|_{W^{1,p}} \leq \|\xi_n\|_{W^{2,p}} \leq \|\Delta\xi_n\|_{L^p} \leq \|h_n - h_{ex}\|_{L^p} + \|h_{ex}\|_{L^p} \leq C. \tag{3.4}$$

By Sobolev embedding, this implies

$$\|A_n\|_{L^q} \leq C, \quad \forall 1 < q < \infty. \tag{3.5}$$

By (3.1), (3.3) and (3.5), we conclude

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 &\leq 2\int_\Omega |\nabla_A u|^2 + |A_n|^2|u_n|^2 \\ &\leq C + \left(\int_\Omega |A_n|^4\right)^{\frac{1}{2}}\left(\int_\Omega |u_n|^4\right)^{\frac{1}{2}} \leq C.\end{aligned} \tag{3.6}$$

From (3.4) and (3.6), subject to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H^1, \\ A_n &\rightharpoonup A \quad \text{in } W^{1,p}, \quad \forall 1 \le p < 2 \end{aligned}$$

and

$$h_n - h_{ex} \rightharpoonup h - h_{ex} \quad \text{in } L^p \quad \forall 1 \le p < 2.$$

In particular, $|u| = 1$ on $\partial\Omega$ by the trace theorem. Since

$$\left\| \frac{h_n - h_{ex}}{\sqrt{\rho_n^2 + \frac{1}{k^2}}} \right\|_{L^2} \le C,$$

and

$$\begin{aligned} h_n - h_{ex} &= \frac{h_n - h_{ex}}{\sqrt{\rho_n^2 + \frac{1}{k^2}}} \cdot \sqrt{\rho_n^2 + \frac{1}{k^2}}, \\ \sqrt{\rho_n^2 + \frac{1}{k^2}} &\longrightarrow \sqrt{\rho^2 + \frac{1}{k^2}} \quad \text{in } L^p, \quad \forall 1 < p < \infty, \end{aligned}$$

we conclude

$$\frac{h_n - h_{ex}}{\sqrt{\rho_n^2 + \frac{1}{k^2}}} \rightharpoonup \frac{h - h_{ex}}{\sqrt{\rho^2 + \frac{1}{k^2}}} \quad \text{in } L^2.$$

It now follows from lower semicontinuity of the functional that (u, A) is a minimizer of $G_k(u, A)$ in $H^1 \times W$. In particular, (u, A) satisfies the Euler–Lagrange equation

$$j_A(u) = \frac{\mu^2}{4} \nabla^\perp \left(\frac{\text{curl } A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right). \tag{3.7}$$

Since

$$\left\| \frac{j_A(u)}{\rho} \right\|_{L^2} \le \|\nabla_A u\|_{L^2} \le C,$$

(3.7) implies

$$\left\| \frac{1}{\rho} \nabla^\perp \left(\frac{\text{curl } A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right) \right\|_{L^2} \le C.$$

It then follows for all $1 \le p < 2$,

$$\begin{aligned} \left\| \nabla \left(\frac{\text{curl } A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right) \right\|_{L^p} &\le \left\| \frac{1}{\rho} \nabla^\perp \left(\frac{\text{curl } A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right) \right\|_{L^2} \cdot \|\rho\|_{L^{\frac{2p}{2-p}}} \\ &\le C. \end{aligned}$$

Given $h = h_{ex}$ on $\partial\Omega$, we conclude

$$\left\| \frac{\text{curl } A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right\|_{L^p} \le C \left\| \nabla \left(\frac{\text{curl } A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right) \right\|_{L^p} \le C.$$

By Sobolev embedding,

$$\left\| \frac{\operatorname{curl} A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right\|_{L^q} \leq C \left\| \frac{\operatorname{curl} A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right\|_{W^{1,p}} \leq C \quad \text{for } \forall 1 \leq q < \infty.$$

Pick $q > 2$,

$$\| h - h_{ex} \|_{L^2} \leq \left\| \frac{\operatorname{curl} A - h_{ex}}{\rho^2 + \frac{1}{k^2}} \right\|_{L^q} \left\| \rho^2 + \frac{1}{k^2} \right\|_{L^{\frac{2q}{q-2}}}.$$

This implies

$$\| \nabla A \|_{L^2} \leq \| \operatorname{curl} A - h_{ex} \|_{L^2} + \| h_{ex} \|_{L^2} \leq C$$

i.e., $(u, A) \in H^1 \times H^1 \subset H^1 \times W$. Since (u, A) is a minimizer in $H^1 \times W$, therefore, a minimizer in $H^1 \times H^1$. Since $|u| = 1$ on $\partial\Omega$, $(u, A) \in V$.

Step 2 Let (u_n, A_n) be a minimizing sequence for $G(u, A)$ in V and (v_n, B_n) are minimizers of $G_n(u, A)$ in V. Therefore,

$$G(u_n, A_n) \geq G_n\left(u_{n,} A_n\right) \geq G_n\left(v_{n,} B_n\right)$$

and (v_n, B_n) satisfies

$$-\frac{\mu^2}{4} \frac{|\operatorname{curl} B_n - h_{ex}|^2}{\left(\rho_n^2 + \frac{1}{n^2}\right)^2} u = \nabla^2_{B_n} u_n + \frac{1}{\varepsilon^2} u_n \left(1 - |u_n|^2\right)\left(3 |u_n|^2 - 1\right) \tag{3.8}$$

$$j_{B_n}(v_n) = \frac{\mu^2}{4} \nabla^{\perp} \left(\frac{\operatorname{curl} B_n - h_{ex}}{\rho_n^2 + \frac{1}{n^2}} \right) \tag{3.9}$$

for $\rho_n = |v_n|$. By estimates from Lemma 2.2, we deduce (v_n, B_n) is a bounded sequence in $H^1 \times H^1$. Subject to a subsequence, we can assume

$$(v_n, B_n) \rightharpoonup (u, A) \quad \text{in } H^1 \times H^1$$

and $|v| = 1$ on $\partial\Omega$. In particular,

$$\frac{\operatorname{curl} B_n - h_{ex}}{\rho_n^2 + \frac{1}{n^2}} \rightharpoonup \frac{\operatorname{curl} A - h_{ex}}{\rho^2} \quad \text{in } L^2.$$

It then follows from lower semicontinuity that

$$\inf_V G(u, A) = \lim G(u_n, A_n) \geq \liminf G_n\left(v_{n,} B_n\right) \geq G(u, A).$$

Therefore, (u, A) is a minimizer of G_{csh} in V. Passing to the limit in (3.8) and (3.9), we find that (u, A) satisfies the Euler–Lagrange equation (1.8)–(1.9).

Finally, we prove $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. Since (u, A) is a minimizer of G_{csh} in V and $A \cdot \nu = 0$ on $\partial\Omega$, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi \, ds = 0 \quad \text{for all } \varphi \text{ that } |u + \varphi| = 1 \text{ on } \partial\Omega.$$

Choosing $\varphi = -2u$, this implies $\int_{\partial\Omega} \frac{\partial u}{\partial \nu} u ds = 0$. It then follows

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi \, ds = 0 \quad \text{for all } \varphi \text{ that } |\varphi| = 1 \text{ on } \partial\Omega.$$

From this, we conclude $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. □

3.2 Energy splitting

This section is devoted to the proof of Theorem 1.6. The proof is done in several steps.

3.2.1 Basic energy estimates

In this section, we present some basic energy estimates on a minimizer (u, A). Recall that, for a suitable choice of gauge, div $A = 0$, and there is a function $\xi \in H^2(\Omega, \mathbb{R})$ such that $A = \nabla^{\perp}\xi = (-\xi_{x_2}, \xi_{x_1})$ with $\xi = 0$ on $\partial\Omega$. Thus,

$$h = \mathrm{curl} A = \Delta\xi, \tag{3.10}$$

and $|\nabla_A u|^2 = |\nabla u|^2 - 2(iu, \xi_{x_2}u_{x_1} - \xi_{x_1}u_{x_2}) + |u|^2 |A|^2$. We introduce ξ_0 as the solution of the following equation

$$\begin{cases} -\frac{\mu_\varepsilon^2}{4}\Delta^2\xi_0 + \Delta\xi_0 = 0 & \text{in } \Omega, \\ \Delta\xi_0 = 1 & \text{on } \partial\Omega, \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

We decompose ξ as

$$\xi = h_{ex}\xi_0 + \zeta \tag{3.12}$$

so that $\zeta = \Delta\zeta = 0$ on $\partial\Omega$. We have the following estimate on ξ_0 and ξ.

Lemma 3.2

$$\begin{cases} -\frac{\mu_\varepsilon^2}{4}\Delta\xi_0 + \xi_0 = -\frac{\mu_\varepsilon^2}{4} & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

with

$$-\frac{\mu_\varepsilon^2}{4} < \xi_0 < 0 \quad \text{in } \Omega,$$
$$0 < \Delta\xi_0 < 1 \quad \text{in } \Omega,$$
$$|\nabla\xi_0|_\infty \le C.$$

Moreover, when $\mu_\varepsilon \longrightarrow 0$,

$$\sup_{\Omega} |\xi_0| = \frac{\mu_\varepsilon^2}{4}(1 - o(1)). \tag{3.14}$$

Proof (3.13) Follows easily from (3.11). In addition, the maximum principle applied to (3.11) yields that $0 < \Delta\xi_0 < 1$ in Ω and then $-\frac{\mu_\varepsilon^2}{4} < \xi_0 < 0$ in Ω.

In order to establish the gradient bound, we note $\|\Delta\xi_0\|_{L^p} \le \|1\|_{L^p} \le C_p$ for all $p < +\infty$. Therefore, $\|\xi_0\|_{W^{2,p}} \le C$ and hence $|\nabla\xi_0|_\infty \le C$ by Sobolev embedding once we fix $p > 2$.

Consider $\frac{\mu^2}{4}f(y) = \xi_0(x)$ where $y = \frac{2}{\mu}x$, then

$$\begin{cases} -\Delta f + f = -1 & \text{in } \Omega_\varepsilon \\ f = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{3.15}$$

and $0 < \Delta f < 1, -1 < f < 0, |\nabla f| \leq \frac{C}{\mu_\varepsilon}$. Since Ω_ε approaches infinity when $\mu_\varepsilon \longrightarrow 0$, we claim $f \simeq -1$ except near boundary. In fact, let $B_{2R} \subset \Omega$, integrating (3.15) over B_t, then integrating t from R to $2R$

$$\begin{aligned}\int_R^{2R}\int_{B_t}(f+1) &= \int_R^{2R}\int_{\partial B_t}\frac{\partial f}{\partial \nu} = c\left|\int_{S^1}\int_R^{2R} tf_t\left(te^{i\theta}\right)dtd\theta\right| \\ &\leq c\left|\int_{S^1}\left[2Rf\left(2Re^{i\theta}\right) - Rf\left(Re^{i\theta}\right)\right]d\theta\right| + \frac{c}{R}\left|\int_{S^1}\int_R^{2R} f\left(te^{i\theta}\right)tdtd\theta\right| \\ &\leq CR.\end{aligned}$$

Since $f+1 > 0$, it follows that $R\int_{B_R}(f+1) \leq \int_R^{2R}\int_{B_t}(f+1) \leq CR$. This implies $\int_{B_R}(f+1) \leq C$, hence

$$K_\varepsilon = \left|\left\{x : x \in B_R \text{ and } f(x) > -1+\sqrt{\mu_\varepsilon}\right\}\right| \leq C\frac{1}{\sqrt{\mu_\varepsilon}}.$$

Choosing $R = \mu_\varepsilon^{-\frac{3}{4}} < \mu_\varepsilon^{-1}$, then $\frac{K_\varepsilon}{|B_R|} \longrightarrow 0$ as $\mu_\varepsilon \longrightarrow 0$. Therefore, $\min f(x) = -1+o(1)$ and (3.14) follows. □

Lemma 3.3 *For* $(\widetilde{u}, \widetilde{A})$ *minimizing* G_{csh} *in* V, *we have* $G_{\text{csh}}(\widetilde{u}, \widetilde{A}) \leq C\mu_\varepsilon^2 h_{ex}^2$ *and*

$$\int_\Omega \left|\nabla_{\widetilde{A}}\widetilde{u}\right|^2 \leq C\mu_\varepsilon^2 h_{ex}^2,$$

$$\int_\Omega \left(1-|\widetilde{u}|^2\right)^2 \leq C\varepsilon\mu_\varepsilon^2 h_{ex}^2, \tag{3.16}$$

$$\left\|\nabla\widetilde{\xi}\right\|_{L^\infty(\Omega)} \leq \frac{Ch_{ex}}{\mu_\varepsilon} + Ch_{ex}. \tag{3.17}$$

Proof As $(\widetilde{u}, \widetilde{A})$ is minimizing, we have

$$G_{\text{csh}}\left(\widetilde{u}, \widetilde{A}\right) \leq G_{\text{csh}}(1, 0) = \frac{\mu_\varepsilon^2}{8}h_{ex}^2|\Omega|$$

and $\int_\Omega |\nabla_{\widetilde{A}}\widetilde{u}|^2 \leq G_{\text{csh}}(\widetilde{u}, \widetilde{A}) \leq C\mu_\varepsilon^2 h_{ex}^2$. To prove (3.16), recall

$$\frac{1}{2}\int_\Omega |\nabla|\widetilde{u}||^2 + \frac{1}{\varepsilon^2}|\widetilde{u}|^2\left(1-|\widetilde{u}|^2\right)^2 \leq G_{\text{csh}}\left(\widetilde{u}, \widetilde{A}\right) \leq C\mu_\varepsilon^2 h_{ex}^2. \tag{3.18}$$

By Cauchy–Schwartz,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla|\widetilde{u}||^2 + \frac{1}{\varepsilon^2}\left(1-|\widetilde{u}|^2\right)^2|\widetilde{u}|^2 &\geq \frac{1}{\varepsilon}\int_\Omega |\nabla|\widetilde{u}||\cdot\left|1-|\widetilde{u}|^2\right||\widetilde{u}| \\ &\geq \frac{1}{4\varepsilon}\int_\Omega \left|\nabla\left(\left(1-|\widetilde{u}|^2\right)^2\right)\right|.\end{aligned} \tag{3.19}$$

Since $|\widetilde{u}| = 1$ on $\partial\Omega$, (3.18) and (3.19) together with Sobolev embedding imply $\int_\Omega(1-|\widetilde{u}|^2)^2 \leq C\varepsilon\mu_\varepsilon^2 h_{ex}^2$. Finally, the gradient estimate on $\widetilde{\xi}$ follows from Lemma 2.2. □

3.2.2 Splitting of energy

From now on, we assume $h_{ex} \leq C\frac{|\log\varepsilon|}{\mu_\varepsilon^2}$. Let $\widetilde{\Omega} = \Omega \setminus \cup_{i\in I} B_i$, where $\{B_i\}_{i\in I}$ is the family of balls given by Proposition 2.13. We have the following identity.

Lemma 3.4

$$\frac{1}{2}\int_\Omega |\nabla_A u|^2 = \int_\Omega \left(\frac{1}{2}\left|\nabla u - i\nabla^\perp \zeta u\right|^2 + \frac{1}{2}h_{ex}^2 |\nabla\xi_0|^2 + h_{ex}\nabla\xi_0\cdot\nabla\zeta\right)$$
$$+2\pi h_{ex}\sum_{i\in I} d_i\xi_0(a_i) + o(1).$$

Proof From (3.10) and (3.12),

$$|\nabla_A u|^2 = \left|\nabla u - i\nabla^\perp\zeta u\right|^2 + h_{ex}^2 |u|^2 |\nabla\xi_0|^2$$
$$+2\left(\nabla u - i\nabla^\perp\zeta u, -ih_{ex}\nabla^\perp\xi_0 u\right).$$

Moreover,

$$\int_\Omega \left(\nabla u - i\nabla^\perp\zeta u, -ih_{ex}\nabla^\perp\xi_0 u\right) = \int_\Omega \left(\nabla u, -ih_{ex}\nabla^\perp\xi_0 u\right) + h_{ex}\int_\Omega |u|^2\,\nabla\xi_0\cdot\nabla\zeta$$

and

$$\left|h_{ex}\int_\Omega \left(1-|u|^2\right)\nabla\xi_0\cdot\nabla\zeta\right| \leq h_{ex}\left(\int_\Omega \left(1-|u|^2\right)^2\right)^{\frac{1}{2}} |\nabla\xi_0|_\infty \left(\int_\Omega |\nabla\zeta|^2\right)^{\frac{1}{2}}$$
$$\leq Ch_{ex}\varepsilon^{\frac{1}{2}}\frac{|\log\varepsilon|}{\mu_\varepsilon} = o(1),$$
$$h_{ex}^2\int_\Omega \left(1-|u|^2\right)|\nabla\xi_0|^2 \leq h_{ex}^2\left(\int_\Omega \left(1-|u|^2\right)^2\right)^{\frac{1}{2}} |\nabla\xi_0|_\infty^2$$
$$\leq Ch_{ex}^2\varepsilon^{\frac{1}{2}}\frac{|\log\varepsilon|}{\mu_\varepsilon} = o(1).$$

The proof is completed using Lemma 3.5. □

Lemma 3.5 *If* $\alpha > 7$ *in Proposition* 2.13*,*

$$\int_\Omega \left(\nabla u, -ih_{ex}\nabla^\perp\xi_0 u\right) = 2\pi h_{ex}\sum_{i\in I} d_i\xi_0(a_i) + o(1).$$

Proof We follow a similar idea as in [20]. First for any $\frac{2(\alpha-1)}{\alpha-4} < p < \infty$,

$$\begin{aligned}
&\left| \int_{\cup B_i} \left(\nabla u, -ih_{ex}\nabla^{\perp}\xi_0 u\right)\right| \\
&\le |\nabla \xi_0|_\infty h_{ex} \left(\int_{\bigcup_i B_i} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\bigcup_i B_i} |u|^2\right)^{\frac{1}{2}} \\
&\le C\frac{1}{\mu_\varepsilon}\left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^2 \|u\|_{L^p(\Omega)} (\mathrm{card} I)^{\frac{1}{2}-\frac{1}{p}}\cdot(\max r_i)^{1-\frac{2}{p}} \\
&= C\left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^{3-\frac{2}{p}}\left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^{-\alpha\left(1-\frac{2}{p}\right)}\frac{1}{\mu_\varepsilon^{2-\frac{2}{p}}} = o(1)\,.
\end{aligned} \tag{3.20}$$

Let $\widetilde{\Omega} = \Omega\backslash \cup B_i$, $v = \frac{u}{|u|}$ and integrating by parts, we get

$$\begin{aligned}
\int_{\widetilde{\Omega}} \left(\nabla u, -ih_{ex}\nabla^{\perp}\xi_0 u\right) &= h_{ex}\int_{\widetilde{\Omega}} \left(iu, (\xi_0)_{x_2} u_{x_1} - (\xi_0)_{x_1} u_{x_2}\right) \\
&= h_{ex}\int_{\widetilde{\Omega}} (iv, dv\wedge d\xi_0) + o(1) \\
&= \sum_{i\in I} h_{ex}\int_{\partial B_i} \xi_0\left(iv, \frac{\partial v}{\partial\tau}\right) + o(1).
\end{aligned} \tag{3.21}$$

We claim that for any $i \in \mathcal{J} = \{i\in I : \overline{B_i}\subset\Omega\}$,

$$h_{ex}\int_{\partial B_i} \xi_0\left(iv, \frac{\partial v}{\partial\tau}\right) = 2\pi h_{ex} d_i \xi_0(a_i) + o\left(\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^2\right). \tag{3.22}$$

To prove this claim, let

$$U_i = \left\{x\in B_i : |u| \le \frac{1}{4}\right\}.$$

Then U_i doesn't intersect ∂B_i and by Stokes' Theorem

$$\begin{aligned}
&\left| \int_{\partial B_i} (\xi_0 - \xi_0(a_i))\left(iv, \frac{\partial v}{\partial\tau}\right) - \int_{\partial U_i} (\xi_0 - \xi_0(a_i))\left(iv, \frac{\partial v}{\partial\tau}\right)\right| \\
&= \left|\int_{B_i\backslash U_i} d\xi_0\wedge\left(iv, \frac{\partial v}{\partial\tau}\right)\right| \le C|\nabla\xi_0|_\infty r_i \left(\int_{B_i\backslash U_i} |\nabla v|^2\right)^{\frac{1}{2}} \\
&\le C\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^{\alpha}\frac{1}{\mu_\varepsilon}\frac{|\log\varepsilon|}{\mu_\varepsilon} = C\frac{1}{|\log\varepsilon|}\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^{\alpha-2}.
\end{aligned}$$

On the other hand, since $|u| = \frac{1}{4}$ on ∂U_i,

$$h_{ex}\int_{\partial U_i}(\xi_0-\xi_0(a_i))\left(iv,\frac{\partial v}{\partial\tau}\right)=16h_{ex}\int_{U_i}(d\xi_0\wedge(iu,du)+(\xi_0-\xi_0(a_i))(idu,du)).$$

Therefore, for $\frac{2(\alpha-1)}{\alpha-6}<p<\infty$,

$$\begin{aligned}
&\left|h_{ex}\int_{\partial U_i}(\xi_0-\xi_0(a_i))\left(iv,\frac{\partial v}{\partial\tau}\right)\right|\\
&\le 16\left|h_{ex}\int_{U_i}d\xi_0\wedge(iu,du)\right|+16\left|h_{ex}\int_{U_i}(\xi_0-\xi_0(a_i))(idu,du)\right|\\
&\le Ch_{ex}|\nabla\xi_0|_\infty\int_{U_i}|udu|+Ch_{ex}r_i|\nabla\xi_0|_\infty\int_{U_i}|\nabla u|^2\\
&\le C\frac{|\log\varepsilon|}{\mu_\varepsilon^2}\left(\int_{U_i}|\nabla u|^2\right)^{\frac12}\left(\int_{U_i}\left|u^2\right|\right)^{\frac12}+C\frac{|\log\varepsilon|}{\mu_\varepsilon^2}\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^{\alpha}\frac{1}{\mu_\varepsilon}\left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^2\\
&\le C\frac{|\log\varepsilon|}{\mu_\varepsilon^2}\frac{|\log\varepsilon|}{\mu_\varepsilon}\|u\|_{L^p(\Omega)}r_i^{1-\frac2p}+C\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^{\alpha-5}\frac{1}{|\log\varepsilon|^2}=o\left(\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^2\right).
\end{aligned}$$

We next deal with balls that intersect $\partial\Omega$. We claim that for all $i\in I\backslash\mathcal{J}$,

$$h_{ex}\int_{\partial B_i\cap\Omega}\xi_0(iv,dv)=o\left(\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^2\right). \tag{3.23}$$

The proof is very similar to that of (3.22). Since $\xi_0=0$ on $\partial\Omega$, letting $U_i=B_i\cap\{|u|\le\frac14\}$

$$\begin{aligned}
h_{ex}\int_{\partial B_i\cap\Omega}\xi_0(iv,dv)&=16h_{ex}\int_{U_i\cap\Omega}d\xi_0\wedge(iu,du)+o\left(\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^2\right)\\
&\le Ch_{ex}|\nabla\xi_0|_\infty\int_{U_i}|udu|+o\left(\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^2\right)\\
&\le C\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^{\alpha\left(1-\frac2p\right)-4+\frac2p}\frac{1}{|\log\varepsilon|^{2-\frac2p}}+o\left(\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^2\right)\\
&=o\left(\left(\frac{\mu_\varepsilon}{|\log\varepsilon|}\right)^2\right)
\end{aligned}$$

for $\frac{2(\alpha-1)}{\alpha-6}<p<\infty$. Using (3.20)–(3.23), and the fact that card $I\le C\left(\frac{|\log\varepsilon|}{\mu_\varepsilon}\right)^2$, the conclusion of the lemma follows. □

Lemma 3.6 *Let* $G_0 = \int_\Omega \frac{h_{ex}^2}{2} |\nabla \xi_0|^2 + \frac{\mu_\varepsilon^2}{8} h_{ex}^2 |\Delta \xi_0 - 1|^2$. *We have the following identity*:

$$\frac{\mu_\varepsilon^2}{8} \int_\Omega |h - h_{ex}|^2 + \frac{1}{2} \int_\Omega h_{ex}^2 |\nabla \xi_0|^2 + \int_\Omega h_{ex} \nabla \xi_0 \cdot \nabla \zeta = G_0 + \frac{\mu_\varepsilon^2}{8} \int_\Omega |\Delta \zeta|^2 .$$

Proof The proof of this lemma is similar to the proof of Lemma 2.4 in [20]. □

A direct corollary of the previous lemma is the following:

Lemma 3.7

$$\begin{aligned} &\frac{\mu_\varepsilon^2}{8} \int_\Omega \frac{|h - h_{ex}|^2}{|u|^2} + \frac{1}{2} \int_\Omega h_{ex}^2 |\nabla \xi_0|^2 + \int_\Omega h_{ex} \nabla \xi_0 \cdot \nabla \zeta \\ &= G_0 + \frac{\mu_\varepsilon^2}{8} \int_\Omega |\Delta \zeta|^2 + o(1) . \end{aligned}$$

Proof We write

$$\frac{\mu_\varepsilon^2}{8} \int_\Omega \frac{|h - h_{ex}|^2}{|u|^2} = \frac{\mu_\varepsilon^2}{8} \int_\Omega |h - h_{ex}|^2 + \frac{\mu_\varepsilon^2}{8} \int_\Omega \frac{|h - h_{ex}|^2}{|u|^2} \left(1 - |u|^2\right) .$$

We can bound

$$\frac{\mu_\varepsilon^2}{8} \int_\Omega \frac{|h - h_{ex}|^2}{|u|^2} \left(1 - |u|^2\right) \leq \frac{\mu_\varepsilon^2}{8} \left\| \frac{h - h_{ex}}{|u|^2} \right\|_{L^p}^2 \left\| |u| \left(1 - |u|^2\right) \right\|_{L^2} \|u\|_{L^q}$$

with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Using $\| \frac{h - h_{ex}}{|u|^2} \|_{L^p} \leq \frac{C_p}{\mu_\varepsilon^2} \sqrt{M_\varepsilon}$, $\|u\|_{L^q} \leq C_q$, and $\|(1 - |u|^2)|u|\|_{L^2} \leq \varepsilon \sqrt{M_\varepsilon}$, the conclusion then follows from Lemma 3.6. □

Lemma 3.8

$$\int_\Omega \left| \nabla u - i u \nabla^\perp \zeta \right|^2 \geq \sum_{i \in I} \int_{B_i} |\nabla u|^2 + o(1) .$$

Proof

$$\begin{aligned} \int_\Omega \left| \nabla u - i u \nabla^\perp \zeta \right|^2 &\geq \int_{\cup_{i \in I} B_i} \left| \nabla u - i u \nabla^\perp \zeta \right|^2 \\ &= \int_{\cup_{i \in I} B_i} |\nabla u|^2 + |u|^2 \left| \nabla^\perp \zeta \right|^2 - 2 \left(\nabla u, i u \nabla^\perp \zeta \right) . \end{aligned}$$

By Lemma 2.2, $|\nabla\zeta|_\infty \leq |A|_{L^\infty} + h_{ex}\,|\xi_0|_{L^\infty} \leq C\frac{|\log\varepsilon|}{\mu_\varepsilon^3}$; therefore,

$$\begin{aligned}\left|\int\limits_{\cup_{i\in I}B_i}\left(\nabla u, iu\nabla^\perp\zeta\right)\right| &\leq |\nabla\zeta|_\infty\left(\int\limits_{\cup_{i\in I}B_i}|\nabla u|^2\right)^{\frac{1}{2}}\left(\int\limits_{\cup_{i\in I}B_i}|u|^2\right)^{\frac{1}{2}}\\ &\leq C\frac{|\log\varepsilon|}{\mu_\varepsilon^3}\frac{|\log\varepsilon|}{\mu_\varepsilon}\|u\|_{L^p(\Omega)}\,r_i^{1-\frac{2}{p}}\,(\mathrm{card}I)^{\frac{1}{2}-\frac{1}{p}}\\ &\leq C\left|\frac{\mu_\varepsilon}{\log\varepsilon}\right|^{(\alpha-1)\left(1-\frac{2}{p}\right)-4+\frac{2}{p}}\frac{1}{|\log\varepsilon|^{3-\frac{2}{p}}} = o(1),\end{aligned}$$

for $p > \frac{2\alpha}{\alpha-5}$, the conclusion follows. □

From this lemma, we deduce that

$$\begin{aligned}&\frac{1}{2}\int\limits_\Omega\left|\nabla u - iu\nabla^\perp\zeta\right|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2\\ &\geq \frac{1}{2}\sum_{i\in I}\int\limits_{B_i}|\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2 + o(1)\\ &= \sum_{i\in I}E_{\mathrm{csh}}(u, B_i),\end{aligned}\tag{3.24}$$

and the last term is bounded below by (2.39).

Proof of Theorem 1.6 We deduce from Lemmas 3.4, 3.6, 3.8 and (2.39), that

$$\begin{aligned}G_{\mathrm{csh}}(u, A) \geq G_0 + 2\pi h_{ex}\sum_{i\in I}d_i\xi_0(a_i)\\ +\pi\sum_{i\in I}|d_i|\left(|\log\varepsilon| - O\left(\log\left|\frac{|\log\varepsilon|}{\mu_\varepsilon}\right|\right)\right).\end{aligned}$$

Since (u, A) is a global minimizer, $G_{\mathrm{csh}}(u, A) \leq G_0$, thus

$$\begin{aligned}\pi\sum_{i\in I}|d_i|\left(|\log\varepsilon| - O\left(\log\left|\frac{\log\varepsilon}{\mu_\varepsilon}\right|\right)\right) &\leq -2\pi h_{ex}\sum_{i\in I}d_i\xi_0(a_i)\\ &\leq 2\pi h_{ex}\sum_{i\in I}|d_i|\max|\xi_0|.\end{aligned}$$

Assume $\mu_\varepsilon \gg e^{-|\log\varepsilon|^\alpha}$ for $0 < \alpha < 1$, then $|\log\mu_\varepsilon| = o(|\log\varepsilon|)$. If $\sum_{i\in I}|d_i| \neq 0$, we deduce that

$$h_{ex} \geq \frac{1}{2\max|\xi_0|}\left(|\log\varepsilon| - O\left(\log\left|\frac{\log\varepsilon}{\mu_\varepsilon}\right|\right)\right) := h_{c_1} \approx \frac{2|\log\varepsilon|}{\mu_\varepsilon^2}.$$

Consequently, if $h_{ex} < h_{c_1}$, we must have $d_i = 0$ for each $i \in I$. Then from Lemmas 3.4, 3.6, 3.8 and (3.24) that

$$G_0 \geq G_{\mathrm{csh}}(u, A) \geq G_0 + \sum_{i\in I}E_{\mathrm{csh}}(u, B_i) + o(1).$$

Therefore,

$$\sum_{i\in I} E_{\text{csh}}(u, B_i) \le o(1). \tag{3.25}$$

We conclude that $|u| \ge \frac{1}{4}$ in B_i. Otherwise since $|u| \ge \frac{1}{2}$ on ∂B_i, if $|u(x_0)| < \frac{1}{4}$ for some $x_0 \in B_i$. Then there is point $x_1 \in B_i$ such that $|u(x_1)| = \frac{3}{8}$ by continuity of u. In particular, we can find a point $Q \in \partial B_i$ such that $\operatorname{dist}(x_1, \partial B_i) = \operatorname{dist}(x_1, Q)$. From this,

$$\frac{1}{2} - \frac{3}{8} \le |u(Q)| - |u(x_1)| \le |\nabla u|_\infty |x_1 - Q| \le \frac{C_0}{\varepsilon} |x_1 - Q|.$$

Therefore, $B_{\frac{1}{8C_0}\varepsilon}(x_1) \subset B_i$ and a similar argument as in Lemma 2.6 implies there is a constant γ_0 that

$$E_{\text{csh}}(u, B_i) \ge E_{\text{csh}}\left(u, B_{\frac{1}{8C_0}\varepsilon}(x_1)\right) \ge \gamma_0,$$

which contradicts (3.25).

We now finish the proof of the critical field strength. We show that if $h_{ex} > \frac{2|\log \varepsilon|}{\mu_\varepsilon^2}$ then there must be a vortex. We prove this by contradiction. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer with $\sum_{j\in J} |d_j| = 0$, then from argument above, we have $|u| \ge \frac{1}{4}$. We claim $G_{\text{csh}}(u_\varepsilon, A_\varepsilon) \ge G_{\text{csh}}\left(1, h_{ex}\nabla^\perp \xi_0\right) + o(1)$.

We write $A_\varepsilon = h_{ex}\nabla^\perp \xi_0 + \nabla^\perp \zeta$. From Lemma 2.2 and energy splitting Lemmas 3.4, 3.7, we have

$$\begin{aligned} G_{\text{csh}}(u_\varepsilon, A_\varepsilon) \ge{}& \frac{1}{2}\int_\Omega \left|\nabla u_\varepsilon - i u_\varepsilon \nabla^\perp \zeta\right|^2 + \frac{1}{\varepsilon^2}|u_\varepsilon|^2\left(1 - |u_\varepsilon|^2\right)^2 + G_0 \\ & + 2\pi h_{ex}\sum_{i\in I} d_i \xi_0(a_i) + \frac{\mu_\varepsilon^2}{8}\int_\Omega |\Delta\zeta|^2 + o(1). \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2}\int_\Omega \left|\nabla u_\varepsilon - i u_\varepsilon \nabla^\perp \zeta\right|^2 &= \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 - 2j(u_\varepsilon)\cdot\nabla^\perp\zeta + |\nabla\zeta|^2 + \frac{1}{2}\int_\Omega |\nabla\zeta|^2\left(|u_\varepsilon|^2 - 1\right) \\ &= \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 - 2j(u_\varepsilon)\cdot\nabla^\perp\zeta + |\nabla\zeta|^2 + o(1) \end{aligned}$$

and $\sum_{j\in J} |d_j| = 0$, by a similar argument as in proof of Lemma 3.5, we conclude

$$\int_\Omega j(u_\varepsilon)\cdot\nabla^\perp\zeta = o(1).$$

Therefore,

$$G_{\text{csh}}(u_\varepsilon, A_\varepsilon) \ge E_{\text{csh}}(u_\varepsilon) + G_0 + \frac{\mu_\varepsilon^2}{8}\int_\Omega |\Delta\zeta|^2 + \frac{1}{2}\int_\Omega |\nabla\zeta|^2 + o(1).$$

Since $(u_\varepsilon, A_\varepsilon)$ is a minimizer, we have $G_{\text{csh}}(u_\varepsilon, A_\varepsilon) \le G_0$; therefore, $E_{\text{csh}}(u_\varepsilon) + \frac{\mu_\varepsilon^2}{8}\int_\Omega |\Delta\zeta|^2 + \frac{1}{2}\int_\Omega |\nabla\zeta|^2 = o(1)$. Therefore,

$$G_{\text{csh}}(u_\varepsilon, A_\varepsilon) \ge G_0 + o(1).$$

We now construct a sequence of functions (u, A) which have lower energy than Meissner energy when $h_{ex} > \frac{2|\log \varepsilon|}{\mu_\varepsilon^2}$. Fix point $a \in \Omega$ such that $|\xi_0(a)| = \sup_\Omega |\xi_0| = \frac{\mu_\varepsilon^2}{4}(1 - o(1))$. Let $A_0 = h_{ex}\nabla^\perp \xi_0 + \nabla^\perp \zeta$, where ξ_0 is defined by (3.11) and ζ satisfies

$$-\frac{\mu_\varepsilon^2}{4}\Delta^2\zeta + \Delta\zeta = 2\pi\delta_\varepsilon \quad \text{in } \Omega \qquad (3.26)$$
$$\Delta\zeta = 0 \quad \text{on } \partial\Omega$$

where δ_ε is an approximation of a δ-distribution given by

$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{\pi\varepsilon^2} & |x-a| < \varepsilon \\ 0 & \text{else.} \end{cases}$$

Define $u = \rho e^{i\varphi}$, where ρ is defined by

$$\rho(x) = \begin{cases} 0 & |x-a| < \varepsilon \\ \frac{|x-a|}{\varepsilon} - 1 & \varepsilon < |x-a| < 2\varepsilon \\ 1 & |x-a| > 2\varepsilon \end{cases}$$

and φ is given by $\nabla\varphi = A_0 - \frac{\mu_\varepsilon^2}{4}\nabla^\perp \operatorname{curl} A_0$. We write $h = \operatorname{curl} A_0$. Then for any $B_R \supset \{a\}$, $\int_{\partial B_R} \partial_\tau \varphi = \int_{B_R} -\frac{\mu_\varepsilon^2}{4}\Delta h + h = 2\pi$. A direct calculation shows $E_{\mathrm{csh}}(u) \le \pi \log \frac{\operatorname{dist}(a,\partial\Omega)}{\varepsilon} + C \le \pi |\log\varepsilon| + C$. We define A as the solution of

$$\operatorname{curl} A = \rho(\operatorname{curl} A_0 - h_{ex}) + h_{ex} \ \text{ in } \Omega$$
$$\operatorname{div} A = 0 \quad \text{in } \Omega$$
$$A \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

Then

$$\begin{aligned}
G_{\mathrm{csh}}(u, A) &= \frac{1}{2}\int_\Omega \rho^2|\nabla\varphi - A|^2 + |\nabla\rho|^2 + \frac{1}{2}\int_\Omega \frac{\mu_\varepsilon^2}{4}\frac{|h-h_{ex}|^2}{\rho^2} + \frac{1}{\varepsilon^2}\rho^2\left(1-\rho^2\right)^2 \\
&= E_{\mathrm{csh}}(u) - \int_\Omega j(u)\cdot A + \frac{1}{2}\int_\Omega \rho^2|A|^2 + \frac{\mu_\varepsilon^2}{4}|\operatorname{curl} A_0 - h_{ex}|^2 \\
&= E_{\mathrm{csh}}(u) - \int_\Omega j(u)\cdot A_0 + \frac{1}{2}\int_\Omega |A_0|^2 + \frac{\mu_\varepsilon^2}{4}|\operatorname{curl} A_0 - h_{ex}|^2 + o(1) \\
&= E_{\mathrm{csh}}(u) - \int_\Omega j(u)\cdot\nabla^\perp(h_{ex}\xi_0 + \zeta) \\
&\quad + G_0 + \frac{\mu_\varepsilon^2}{8}\int_\Omega |\Delta\zeta|^2 + \frac{1}{2}\int_\Omega |\nabla\zeta|^2 + o(1) \\
&\le \pi|\log\varepsilon| + C + G_0 - 2\pi h_{ex}\frac{\mu_\varepsilon^2}{4} \\
&\quad + \frac{\mu_\varepsilon^2}{8}\int_\Omega |\Delta\zeta|^2 + \frac{1}{2}\int_\Omega |\nabla\zeta|^2 + 2\pi\zeta(a).
\end{aligned}$$

The last step follows from similar argument as in Lemmas 3.5 and 3.7. Multiplying (3.26) by ζ and integrating over Ω shows $\frac{\mu_\varepsilon^2}{4}\int_\Omega |\Delta\zeta|^2 + \int_\Omega |\nabla\zeta|^2 = -2\pi\zeta(a)$; hence,

$$\frac{\mu_\varepsilon^2}{8}\int_\Omega |\Delta\zeta|^2 + \frac{1}{2}\int_\Omega |\nabla\zeta|^2 + 2\pi\zeta(a) < 0$$

and we have

$$G_{\text{csh}}(u, A) \le G_0 + \pi|\log\varepsilon| - 2\pi h_{ex}\frac{\mu_\varepsilon^2}{4} + C.$$

When $h_{ex} > \frac{2|\log\varepsilon|}{\mu_\varepsilon^2}$, there exists $\delta > 0$ such that

$$\pi|\log\varepsilon| - 2\pi h_{ex}\frac{\mu_\varepsilon^2}{4} + C < C - \delta|\log\varepsilon| < -\frac{|C|}{2}.$$

Thus

$$G_{\text{csh}}(u, A) < G_0 - \frac{|C|}{2} \le G_{\text{csh}}(u_\varepsilon, A_\varepsilon).$$

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