Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 336 (2007) 372–398
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# On the singular set for Lipschitzian critical points of polyconvex functionals

Sungwon Cho, Xiaodong Yan *

*Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*

Received 2 January 2007
Available online 3 March 2007
Submitted by H.R. Parks

**Abstract**

Partial regularity is proved for Lipschitzian critical points of polyconvex functionals provided $\|Du\|_{L^\infty}$ is small enough. In particular, the singular set for a Lipschitzian critical point has Hausdorff dimension strictly less than $n$ when $\|Du\|_{L^\infty}$ is small enough. Model problems treated include

$$\int_\Omega |\nabla u|^2 + |\det \nabla u|^2,$$

where $u : \Omega (\subset \mathbb{R}^2) \to \mathbb{R}^2$, and

$$\int_\Omega |\nabla u|^2 + |\nabla u|^s + |\mathrm{Ad}\, \nabla u|^s + |\det \nabla u|^s,$$

where $u : \Omega (\subset \mathbb{R}^3) \to \mathbb{R}^3$ with $s \geqslant 2$. Moreover, it is shown that the singular set of a Lipschitzian global minimizer has Hausdorff dimension strictly less than $n$.


*Keywords:* Partial regularity; Hausdorff dimension; Polyconvexity

* Corresponding author.
*E-mail addresses:* cho@math.msu.edu (S. Cho), xiayan@math.msu.edu (X. Yan).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2007.02.078

## 1. Introduction

We study regularity for critical points of the following variational integral

$$I(u) = \int_{\Omega} F^1(Du) + \sum_{i=2}^{k} F^i(\Lambda_i Du) \tag{1.1}$$

where $u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^N$, and $F^i(P)$ are convex functions in $P$ for $i = 1, \ldots, k$.

Regularity for minimizers or critical points of functionals of the form

$$I(u) = \int_{\Omega} F(x, u, Du) \tag{1.2}$$

has been studied extensively in literature. In particular, when $F(x, u, p)$ is convex in $p$, regularity results for minimizers and critical points of (1.2) can be found in many literatures (for example [6,12]). When $N > 1$, natural assumption on $F$ is $F$ being quasiconvex in $p$. Existence theory for minimizers of (1.2) in such cases are well understood (see for example [6,12] and [1]). As for regularity results on minimizers of quasiconvex functionals, a well-known result of Evans [2] shows global minimizers of $\int_{\Omega} F(Du)$ with strongly quasiconvex integrand are smooth outside a closed subset of Lebesgue measure zero. (See [3] and [7] for general cases.) This result was extended by Kristensen and Taheri [10] to the case of strong local minimizers (local with respect to variations in $W^{1,p}$ for $p < \infty$). Recently, Kristensen and Mingione [9] proved, under suitable growth assumptions, that the singular set for Lipschitzian minimizers of strongly quasiconvex functional has Hausdorff dimension strictly less than $n$. We also refer the readers to [8] for a complete treatment for singular set of minimizers in the case of convex integrals. This is in sharp contrast with the regularity results on critical points of quasiconvex functionals. Striking counterexamples constructed by Müller and Šverák [14] show that in general, one cannot expect any partial regularity for a critical point of strongly quasiconvex functionals without further assumptions on the structure of $F$. More precisely, they constructed a smooth strongly quasiconvex functional of the form $\int_{\Omega} F(Du)$, where the second derivative of $F$ is bounded and the corresponding Euler–Lagrange equation admits a Lipschitz solution $u$ yet $u$ is not $C^1$ on any open set. This is extended by Kristensen and Taheri [10] to the case of weak local minimizer (small variations in $W^{1,\infty}$), showing that weak local minimizers can be nowhere $C^1$. When $F$ is rank one convex with bounded second derivative, Moser [13] proved that any Lipschitz critical points with small BMO norm is smooth.

Our model was motivated by nonlinear elasticity. Most of the model problems from nonlinear elasticity involves polyconvex functionals, i.e. a convex function of the various minors of $Du$. Recall that polyconvexity implies quasiconvexity. Since the model problem we study here does not fit into the category of quasiconvex (or rank one convex) functional studied in [2,9,10,13] due to the growth control assumptions, it is interesting to study the regularity problem separately. The model was first introduced by Fusco and Hutchinson [4,5] where they proved partial regularity for global minimizers in Sobolev spaces. The method there relies crucially on the fact that $u$ is a minimizer. It is not known whether any partial regularity can be expected for critical points of this model. In general, one cannot expect partial regularity result for the critical points of strongly polyconvex functionals as shown by Székelyhidi [16]. In the similiar spirit as counterexample by Müller and Šverák [14], Székelyhidi constructed a smooth strongly polyconvex function whose Euler–Lagrange equation has a Lipschitz yet nowhere $C^1$ weak solution. This shows one cannot expect partial regularity for the critical points of polyconvex function without further assumptions on the structure of $F$.

In this paper, we restrict our attention to problems of the form (1.1). Some model problems treated include

$$\int_{\Omega} |Du|^2 + |\det Du|^2$$

where $u : \Omega(\subset \mathbb{R}^2) \to \mathbb{R}^2$, and

$$\int_{\Omega} |Du|^2 + |Du|^s + |\mathrm{Ad}\, Du|^s + |\det Du|^s$$

where $u : \Omega(\subset \mathbb{R}^3) \to \mathbb{R}^3$ with $s \geqslant 2$.

The main result is that Lipschitzian critical points for (1.1) with small $\|Du\|_{L^\infty}$ are $C^{1,\alpha}$ except on a closed set of Hausdorff dimension strictly less than $n$, for each $0 < \alpha < 1$. Higher regularity follows from standard bootstrapping arguments. It is also shown that the singular set of any Lipschitzian minimizers of (1.1) has Hausdorff dimension strictly less than $n$ as a byproduct.

Our main observation is that for critical points with small $\|Du\|_{L^\infty}$, we can prove Caccioppoli type inequality. The proof relies essentially on the fact that determinant is a null Lagrangian. A standard blow up argument then concludes $u$ is smooth away from a set of measure zero. Caccioppoli inequality can be also proved in the minimizer case following Evans' idea [2]. Once we have Caccioppoli inequality, we can follow Mingione and Kristensen's idea [9] to reduce the Hausdorff dimension on singular set.

The paper is organized as follows. In Section 2, we introduce some preliminary notations and state the main results. In Section 3, we prove Caccioppoli inequality for critical points with small $\|Du\|_{L^\infty}$. In Section 4, we prove Lipschitzian critical points with small $\|Du\|_{L^\infty}$ are $C^{1,\alpha}$ except on a closed set of measure zero. In the last section, we prove reduced Hausdorff dimension estimate on singular set for Lipschitzian critical points with sufficiently small Lipschitz norms or any Lipschitzian minimizers.

## 2. Preliminaries and main results

### 2.1. Notations

We use the following standard notation:

$$B_t = \{y \in \mathbb{R}^n\colon\ |y| < t\},$$
$$B(x,r) = \{y \in \mathbb{R}^n\colon\ |y-x| < r\},$$
$$\fint_E f = \frac{1}{|E|}\int_E f,$$
$$(f)_t = \fint_{B_t} f,$$
$$(f)_{x,r} = \fint_{B_{x,r}} f,$$

"$f \rightharpoonup f$ in $L^k$" denotes weak convergence,

"$f \to f$ in $L^k$" denotes strong convergence.

Given $n \times N$ matrix $A$, for $1 \leqslant k \leqslant \min(n, N)$, $\Lambda_k A$ denotes a vector whose coordinates are $k \times k$ minors of $A$, and for two $n \times N$ matrices $A$ and $B$, following symbols in [4] Sections 2.1, 2.2, we write

$$\Lambda_k(A+B) = \Lambda_k A + \sum_{i=1}^{k-1} \Lambda_{k-i} A \odot \Lambda_i B + \Lambda_k B,$$
$$\Lambda_0 A = \Lambda_0 B = 1.$$

## 2.2. *Properties of $\Lambda_k Du$*

Let $u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^N$ be a smooth map, for any $\mu = (\mu_1, \ldots, \mu_k)$, $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have

$$\begin{aligned}(\Lambda_k Du)_{\mu\lambda} &= \det\left[D_{\lambda_i} u^{\mu_j}\right]_{i,j=1}^k \\ &= \sum_{i=1}^k (-1)^{i+j} D_{\lambda_i}\left(u^{\mu_j} (\Lambda_{k-1} Du)_{(\mu_1, \ldots, \widehat{\mu_j}, \ldots, \mu_k)(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k)}\right).\end{aligned} \tag{2.1}$$

Throughout this paper, we abbreviate (2.1) as

$$\Lambda_i Du = \sum D(u \odot \Lambda_{i-1} Du). \tag{2.2}$$

And similarly, for any smooth function $\varphi : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^N$, we use abbreviation

$$\Lambda_{i-1} Du \odot D\varphi = \sum D(\varphi \odot \Lambda_{i-1} Du). \tag{2.3}$$

In particular, all $1 \leqslant j \leqslant i-1$ we can write

$$\Lambda_{i-j} Du \odot \Lambda_j Du = c(i,j) \Lambda_i Du \tag{2.4}$$

with $c(i,j) = \frac{i!}{j!(i-j)!} > 1$. We refer the readers to [4] Sections 2.1, 2.2 for further properties on $\Lambda_k Du$.

## 2.3. *Main results*

Suppose $u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^N$. Consider functionals of the form

$$I(u) = \int_\Omega F^1(Du) + \sum_{i=2}^k F^i(\Lambda_i Du). \tag{2.5}$$

Assume $F^i$ satisfies the following hypotheses:

(H1) $F^i$ is $C^2$ for $i = 1, \ldots, k$;
(H2) $s \geqslant 2$;
(H3) (a) $|D^2 F^1(p)| \leqslant \Gamma(1 + |p|^{s-2})$,
 (b) $D_{p_i^\alpha p_j^\beta} F^1(p) \xi_i^\alpha \xi_j^\beta \geqslant \gamma(1 + |p|^{s-2})|\xi|^2$

for all $\xi \in M^{N \times n}$, some $\Gamma$, $\gamma > 0$;

(H4) For $i = 2, \dots, k$,
(a) $|D^2 F^i(q)| \leqslant \Gamma |q|^{s-2}$,
(b) $D_{q_\alpha q_\beta} F^i(q) \xi_\alpha \xi_\beta \geqslant \gamma |q|^{s-2} |\xi|^2$

for all $\xi \in \mathbb{R}^{q_i}$, some $\Gamma, \gamma > 0$.

**Remark 1.** We remark that for partial regularity for global minimizers in Sobolev spaces [4], one needs $s > n-1$ for $n > 2$.

We say $u$ is a weak solution to Euler–Lagrange equation corresponding to (2.5) if $u$ satisfies

$$0 = \int_\Omega DF^1(Du) D\varphi + \sum_{i=2}^k DF^i(\Lambda_i Du) \Lambda_{i-1} Du \odot D\varphi \tag{2.6}$$

for any $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

The main theorems of this paper are

**Theorem 1.** *There exists a constant $M = M(\Gamma, \gamma, s, n, N, k) > 0$ such that if $u \in W^{1,\infty}$ is a weak solution to* (2.6) *with $\|Du\|_{L^\infty} \leqslant M$, there exist an open set $\Omega_0 \subset \Omega$ and $p = p(n, N, s, \gamma, \Gamma, \|Du\|_{L^\infty}, k) < n$, such that*

$$\mathcal{H}^p(\Omega \sim \Omega_0) = 0, \quad u \in C^{1,\alpha}(\Omega_0)$$

*for all* $0 < \alpha < 1$.

**Theorem 2.** *If $u \in W^{1,\infty}$ is a global minimizer of* (2.5)*, there exist an open set $\Omega_0 \subset \Omega$ and a constant $p = p(n, N, s, \gamma, \Gamma, \|u\|_{W^{1,\infty}}, k) < n$, such that*

$$\mathcal{H}^p(\Omega \sim \Omega_0) = 0, \quad u \in C^{1,\alpha}(\Omega_0)$$

*for all* $0 < \alpha < 1$.

## 3. Caccioppoli inequalities

**Lemma 1.** *There exists a constant $M = M(\Gamma, \gamma, s, n, N, k) > 0$ such that for any constant matrix $A$ satisfying $|A| \leqslant M$, if $u \in W^{1,\infty}$ satisfies*

$$0 = \int_\Omega DF^1(A + \lambda Du) D\varphi + \int_\Omega \sum_{i=2}^k DF^i\big(\Lambda_i(A + \lambda Du)\big) \Lambda_{i-1}(A + \lambda Du) \odot D\varphi \tag{3.1}$$

*for any $\varphi \in C_0^\infty(\Omega)$ and $\lambda \|Du\|_{L^\infty} \leqslant M$, then there exists $c = c(M, s, \gamma, k, \Gamma, n, N)$ such that for any $B_r \Subset \Omega$ and any $a \in \mathbb{R}^N$,*

$$\int_{B_{\frac{r}{2}}} |Du|^2 + \lambda^{s-2} \int_{B_{\frac{r}{2}}} |Du|^s + |A|^{s-2} \int_{B_{\frac{r}{2}}} |Du|^2$$
$$+ \lambda^{-2} \sum_{i=2}^k \int_{B_{\frac{r}{2}}} |\Lambda_i A|^{s-2} \big|\Lambda_i(A + \lambda Du) - \Lambda_i(A)\big|^2$$

$$+\lambda^{-2}\sum_{i=2}^{k}\int_{B_{\frac{r}{2}}}\big|\Lambda_i(A+\lambda Du)-\Lambda_i(A)\big|^s$$
$$\leqslant \frac{c}{r^2}\int_{B_r}|u-a|^2+\frac{c}{r^s}\lambda^{s-2}\int_{B_r}|u-a|^s+\frac{c}{r^2}|A|^{s-2}\int_{B_r}|u-a|^2.$$

**Proof.** Since $u$ is a weak solution of (3.1), using (2.3) and the fact that determinant is a null Lagrangian, we have

$$0=\int_{\Omega}\big[DF^1(A+\lambda Du)-DF^1(A)\big]\cdot D\varphi$$
$$+\int_{\Omega}\sum_{i=2}^{k}\big[DF^i\big(\Lambda_i(A+\lambda Du)\big)-DF^i\big(\Lambda_i(A)\big)\big]\big(\Lambda_{i-1}(A+\lambda Du)\odot D\varphi\big) \tag{3.2}$$

for any $\varphi\in C_0^\infty(\Omega)$. If $u\in W^{1,\infty}(\Omega)$, then (3.2) remains true for any $\varphi\in W_0^{1,s}(\Omega)$ by approximation. For any $B_r\Subset\Omega$, let $\eta\in C_0^1(B_r)$ where $\eta\equiv 1$ on $B_t$ and $|D\eta|\leqslant\frac{c}{r-t}$. We consider

$$\varphi_0=\eta^2(u-a)\in W_0^{1,s}(\Omega),$$

then it follows from (2.3) that

$$0=\int_{\Omega}\big[DF^1(A+\lambda Du)-DF^1(A)\big]\cdot D\varphi_0$$
$$+\int_{\Omega}\sum_{i=2}^{k}\big[DF^i\big(\Lambda_i(A+\lambda Du)\big)-DF^i\big(\Lambda_i(A)\big)\big]\Lambda_{i-1}(A+\lambda Du)\odot D\varphi_0,$$

i.e.

$$\int_{\Omega}\big[DF^1(A+\lambda Du)-DF^1(A)\big]\cdot\eta^2 Du$$
$$+\lambda^{-1}\int_{\Omega}\sum_{i=2}^{k}\eta^2\big[\big[DF^i\big(\Lambda_i(A+\lambda Du)\big)-DF^i\big(\Lambda_i(A)\big)\big]\big(\Lambda_i(A+\lambda Du)-\Lambda_i A\big)\big]$$
$$=-\int_{\Omega}\big[DF^1(A+\lambda Du)-DF^1(A)\big]\cdot 2\eta(u-a)\odot D\eta$$
$$-\int_{\Omega}\sum_{i=2}^{k}\big[\big[DF^i\big(\Lambda_i(A+\lambda Du)\big)-DF^i\big(\Lambda_i(A)\big)\big]$$
$$\cdot\big(\Lambda_{i-1}(A+\lambda Du)\odot 2\eta(u-a)\odot D\eta\big)\big]$$
$$+\int_{\Omega}\sum_{i=2}^{k}\eta^2\big[\big[DF^i\big(\Lambda_i(A+\lambda Du)\big)-DF^i\big(\Lambda_i(A)\big)\big]$$
$$\cdot\big(\lambda^{-1}\big(\Lambda_i(A+\lambda Du)-\Lambda_i A\big)-\Lambda_{i-1}(A+\lambda Du)\odot Du\big)\big]. \tag{3.3}$$

We estimate left side of (3.3) from below

$$L = \int_{\Omega} \bigl[DF^1(A+\lambda Du) - DF^1(A)\bigr] \cdot \eta^2 Du$$
$$+ \lambda^{-1} \int_{\Omega} \sum_{i=2}^{k} \bigl[DF^i\bigl(\Lambda_i(A+\lambda Du)\bigr) - DF^i\bigl(\Lambda_i(A)\bigr)\bigr]\eta^2\bigl(\Lambda_i(A+\lambda Du) - \Lambda_i(A)\bigr).$$

It follows from assumptions (H3) and (H4) that

$$\lambda L = \int_{B_r} \eta^2 \int_0^1 D^2F^1(A+\tau\lambda Du)\, d\tau\, \lambda Du \cdot \lambda Du$$
$$+ \int_{B_r} \eta^2 \sum_{i=2}^{k} \Biggl[\int_0^1 D^2F^i\bigl(\Lambda_i A + \tau\bigl(\Lambda_i(A+\lambda Du) - \Lambda_i A\bigr)\bigr)\, d\tau$$
$$\cdot \bigl(\Lambda_i(A+\lambda Du) - \Lambda_i A\bigr) \cdot \bigl(\Lambda_i(A+\lambda Du) - \Lambda_i A\bigr)\Biggr]$$
$$\geqslant \int_{B_r} \eta^2\gamma \int_0^1 \bigl(1 + |A+\tau\lambda Du|^{s-2}\bigr)\, d\tau\, |\lambda Du|^2$$
$$+ \int_{B_r} \eta^2\gamma \sum_{i=2}^{k} \Biggl[\int_0^1 \bigl|\Lambda_i A + \tau\bigl(\Lambda_i(A+\lambda Du) - \Lambda_i A\bigr)\bigr|^{s-2}\, d\tau$$
$$\cdot \bigl|\Lambda_i(A+\lambda Du) - \Lambda_i A\bigr|^2\Biggr].$$

From [2, Lemma 8.1], there exists $\sigma > 0$ such that for any two $n \times N$ matrices $B$ and $C$,

$$\sigma\bigl(|C|^{s-2} + |B|^{s-2}\bigr) \leqslant \int_0^1 (1-\tau)|C+\tau B|^{s-2}\, d\tau$$
$$\leqslant \int_0^1 |C+\tau B|^{s-2}\, d\tau.$$

Thus

$$\lambda L \geqslant \sigma\gamma \int_{B_r} \eta^2\bigl(1 + |A|^{s-2} + |\lambda Du|^{s-2}\bigr)|\lambda Du|^2$$
$$+ \sigma\gamma \int_{B_r} \eta^2 \sum_{i=2}^{k} \bigl[\bigl(|\Lambda_i A|^{s-2} + \bigl|\Lambda_i(A+\lambda Du) - \Lambda_i A\bigr|^{s-2}\bigr)\bigl|\Lambda_i(A+\lambda Du) - \Lambda_i A\bigr|^2\bigr]$$

$$\geqslant \sigma\gamma \int_{B_r} \eta^2\big(|\lambda Du|^2 + |\lambda Du|^s\big) + \sigma\gamma \int_{B_r} \eta^2 \sum_{i=2}^{k} \big|\Lambda_i(A+\lambda Du) - \Lambda_i A\big|^s$$

$$+ \sigma\gamma \int_{B_r} \eta^2 \sum_{i=2}^{k} |\Lambda_i A|^{s-2} \big|\Lambda_i(A+\lambda Du) - \Lambda_i A\big|^2 + \sigma\gamma \int_{B_r} \eta^2 |A|^{s-2} |\lambda Du|^2. \quad (3.4)$$

For simplicity of notations, if not specified otherwise, $c$ denotes a constant possibly depending on the given parameters $\gamma, \Gamma, n, N, s$ and which may change from line to line. Next we estimate right side of (3.3) from above. By assumptions (H3) and (H4), we have

$$\lambda R = -\int_{B_r} \int_0^1 D^2F^1(A+\tau\lambda Du)\,d\tau\,\lambda Du \cdot 2\eta\lambda(u-a)D\eta$$

$$- \int_{B_r} \sum_{i=2}^{k} \bigg[ \int_0^1 D^2F^i\big(\Lambda_i A + \tau\big(\Lambda_i(A+\lambda Du) - \Lambda_i A\big)\big)\,d\tau$$

$$\cdot \big(\Lambda_i(A+\lambda Du) - \Lambda_i A\big) \cdot \big(\Lambda_{i-1}(A+\lambda Du) \odot 2\eta\lambda(u-a)D\eta\big)\bigg]$$

$$+ \int_{\Omega} \sum_{i=2}^{k} \eta^2 \big[\big[DF^i\big(\Lambda_i(A+\lambda Du)\big) - DF^i\big(\Lambda_i(A)\big)\big]$$

$$\cdot \big(\Lambda_i(A+\lambda Du) - \Lambda_i A - \Lambda_{i-1}(A+\lambda Du) \odot \lambda Du\big)\big]$$

$$\leqslant c(\Gamma) \int_{B_r} \int_0^1 \big(1 + |A+\tau\lambda Du|^{s-2}\big)\,d\tau\,|\lambda Du| \big|\eta\lambda(u-a)D\eta\big|$$

$$+ c(\Gamma) \int_{B_r} \sum_{i=2}^{k} \bigg[ \int_0^1 \big|\Lambda_i A + \tau\big(\Lambda_i(A+\lambda Du) - \Lambda_i A\big)\big|^{s-2}\,d\tau$$

$$\cdot \big|\Lambda_i(A+\lambda Du) - \Lambda_i A\big| \big|\Lambda_{i-1}(A+\lambda Du)\big| \big|\eta\lambda(u-a)D\eta\big|\bigg]$$

$$+ \int_{\Omega} \sum_{i=2}^{k} \eta^2 \bigg[ \int_0^1 D^2F^i\big(\Lambda_i A + \tau\big(\Lambda_i(A+\lambda Du) - \Lambda_i A\big)\big)\,d\tau$$

$$\cdot \big(\Lambda_i(A+\lambda Du) - \Lambda_i A\big)\big(\Lambda_i(A+\lambda Du) - \Lambda_i A - \Lambda_{i-1}(A+\lambda Du) \odot \lambda Du\big)\bigg]$$

$$\leqslant c(\Gamma, s) \int_{B_r \setminus B_t} \big(1 + |A|^{s-2} + |\lambda Du|^{s-2}\big) |\lambda Du| \big|\eta\lambda(u-a)D\eta\big|$$

$$+ c(\Gamma, s) \int_{B_r \setminus B_t} \sum_{i=2}^{k} \big[\big(|\Lambda_i A|^{s-2} + \big|\Lambda_i(A+\lambda Du) - \Lambda_i A\big|^{s-2}\big)$$

$$
\begin{aligned}
&\quad \cdot \big|\eta\lambda(u-a)D\eta\big|\big|\Lambda_i(A+\lambda Du)-\Lambda_i A\big|\big|\Lambda_{i-1}(A+\lambda Du)\big|\big]\\
&\quad + \int\limits_{\Omega}\sum_{i=2}^{k}\eta^2\Bigg[\int\limits_0^1 D^2F^i\big(\Lambda_i A+\tau\big(\Lambda_i(A+\lambda Du)-\Lambda_i A\big)\big)\,d\tau\\
&\quad \cdot\big(\Lambda_i(A+\lambda Du)-\Lambda_i A\big)\\
&\quad \cdot\left(\sum_{j=1}^{i}\Lambda_{i-j}A\odot\Lambda_j(\lambda Du)-\sum_{j=0}^{i-1}\Lambda_{i-1-j}A\odot\Lambda_j(\lambda Du)\odot\lambda Du\right)\Bigg]\\
&= R_1+R_2+R_3. \qquad (3.5)
\end{aligned}
$$

By Cauchy–Schwartz inequality,

$$
\begin{aligned}
R_1 &= c(\Gamma,s)\int\limits_{B_r\setminus B_t}\big(1+|A|^{s-2}+|\lambda Du|^{s-2}\big)|\lambda Du|\big|\eta\lambda(u-a)D\eta\big|\\
&\leqslant c(\Gamma,s)\int\limits_{B_r\setminus B_t}\Bigg[|\lambda Du|^2+|A|^{s-2}|\lambda Du|^2+|\lambda Du|^s+\frac{|\lambda(u-a)|^2}{(r-t)^2}\\
&\quad +|A|^{s-2}\frac{|\lambda(u-a)|^2}{(r-t)^2}+\frac{|\lambda(u-a)|^s}{(r-t)^s}\Bigg] \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
R_2 &= c(\Gamma,s)\int\limits_{B_r\setminus B_t}\sum_{i=2}^{k}\big[\big(|\Lambda_i A|^{s-2}+\big|\Lambda_i(A+\lambda Du)-\Lambda_i A\big|^{s-2}\big)\big|\eta\lambda(u-a)D\eta\big|\\
&\quad \cdot\big|\Lambda_i(A+\lambda Du)-\Lambda_i A\big|\big|\Lambda_{i-1}(A+\lambda Du)\big|\big]\\
&\leqslant c(\Gamma,s)\int\limits_{B_r\setminus B_t}\sum_{i=2}^{k}|\Lambda_i A|^{s-2}\big|\Lambda_i(A+\lambda Du)-\Lambda_i A\big|^2\\
&\quad + c(\Gamma,s)\int\limits_{B_r\setminus B_t}\sum_{i=2}^{k}\big|\Lambda_i(A+\lambda Du)-\Lambda_i A\big|^s\\
&\quad + \frac{c(\Gamma,s,k)\sum_{i=2}^{k}|\Lambda_i A|^{s-2}M^{2(i-1)}}{(r-t)^2}\int\limits_{B_r\setminus B_t}\big|\lambda(u-a)\big|^2\\
&\quad + \frac{c(\Gamma,s,k)\sum_{i=2}^{k}M^{s(i-1)}}{(r-t)^s}\int\limits_{B_r\setminus B_t}\big|\lambda(u-a)\big|^s. \qquad (3.7)
\end{aligned}
$$

By (2.4),

$$
R_3=\int\limits_{\Omega}\sum_{i=2}^{k}\eta^2\Bigg[\int\limits_0^1 D^2F^i\big(\Lambda_i A+\tau\big(\Lambda_i(A+\lambda Du)-\Lambda_i A\big)\big)\,d\tau\big(\Lambda_i(A+\lambda Du)-\Lambda_i A\big)
$$

$$
\begin{aligned}
&\cdot\left(\sum_{j=1}^{i}\Lambda_{i-j}A\odot\Lambda_j(\lambda Du)-\sum_{j=0}^{i-1}\Lambda_{i-1-j}A\odot\Lambda_j(\lambda Du)\odot\lambda Du\right)\Bigg]\\
&\leqslant c(\Gamma,s)\int\limits_{B_r}\sum_{i=2}^{k}\bigg[\left(|\Lambda_i A|^{s-2}+\left|\Lambda_i(\lambda Du+A)-\Lambda_i A\right|^{s-2}\right)\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|\\
&\quad\cdot\eta^2\left|\sum_{j=1}^{i}(j-1)\Lambda_{i-j}A\odot\Lambda_j(\lambda Du)\right|\bigg]\\
&\leqslant\frac{c(\Gamma,s)}{2}\varepsilon\int\limits_{B_r}\sum_{i=2}^{k}\eta^2|\Lambda_i A|^{s-2}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^2\\
&\quad+\frac{c(\Gamma,s)}{2}\varepsilon\int\limits_{B_r}\sum_{i=2}^{k}\eta^2\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^s\\
&\quad+\frac{c(\Gamma,s)}{2\varepsilon}\int\limits_{B_r}\sum_{i=2}^{k}\eta^2|\Lambda_i A|^{s-2}\left|\sum_{j=1}^{i}(j-1)\Lambda_{i-j}A\odot\Lambda_j(\lambda Du)\right|^2\\
&\quad+\frac{c(\Gamma,s)}{2\varepsilon}\int\limits_{B_r}\sum_{i=2}^{k}\eta^2\left|\sum_{j=1}^{i}(j-1)\Lambda_{i-j}A\odot\Lambda_j(\lambda Du)\right|^s\\
&\leqslant\frac{c(\Gamma,s)}{2}\varepsilon\int\limits_{B_r}\sum_{i=2}^{k}\eta^2|\Lambda_i A|^{s-2}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^2\\
&\quad+\frac{c(\Gamma,s)}{2}\varepsilon\int\limits_{B_r}\sum_{i=2}^{k}\eta^2\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^s\\
&\quad+\frac{c(\Gamma,s,k)\sum_{i=2}^{k}(M^{i(s-2)}M^{2(i-1)}+M^{s(i-1)})}{2\varepsilon}\int\limits_{B_r}\eta^2\left(|\lambda Du|^2+|\lambda Du|^s\right).
\end{aligned}\tag{3.8}
$$

(3.5)–(3.8) give

$$
\begin{aligned}
\lambda R&\leqslant c\int\limits_{B_r\setminus B_t}\left[|\lambda Du|^2+|A|^{s-2}|\lambda Du|^2+|\lambda Du|^s+\frac{|\lambda(u-a)|^2}{(r-t)^2}+|A|^{s-2}\frac{|\lambda(u-a)|^2}{(r-t)^2}\right.\\
&\quad\left.+\frac{|\lambda(u-a)|^s}{(r-t)^s}\right]\\
&\quad+c\int\limits_{B_r\setminus B_t}\sum_{i=2}^{k}|\Lambda_i A|^{s-2}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^2\\
&\quad+c\int\limits_{B_r\setminus B_t}\sum_{i=2}^{k}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^s+c\sum_{i=2}^{k}\frac{M^{2(i-1)}|\Lambda_i A|^{s-2}}{(r-t)^2}\int\limits_{B_r\setminus B_t}\left|\lambda(u-a)\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ c\sum_{i=2}^{k}\frac{M^{s(i-1)}}{(r-t)^s}\int_{B_r\setminus B_t}\left|\lambda(u-a)\right|^s \\
&+ \frac{c(\Gamma,s)}{2}\varepsilon\int_{B_r}\sum_{i=2}^{k}\eta^2|\Lambda_i A|^{s-2}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^2 \\
&+ \frac{c(\Gamma,s)}{2}\varepsilon\int_{B_r}\sum_{i=2}^{k}\eta^2\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^s \\
&+ \frac{c(\Gamma,s,k)\sum_{i=2}^{k}(M^{i(s-2)}M^{2(i-1)}+M^{s(i-1)})}{2\varepsilon}\int_{B_r}\eta^2\left(|\lambda Du|^2+|\lambda Du|^s\right). \qquad (3.9)
\end{aligned}
$$

(3.4) and (3.9) give

$$
\begin{aligned}
&\int_{B_r}\eta^2\left(|\lambda Du|^2+|A|^{s-2}|\lambda Du|^2+|\lambda Du|^s\right)+\int_{B_r}\eta^2\sum_{i=2}^{k}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A)\right|^s \\
&\quad+\int_{B_r}\eta^2\sum_{i=2}^{k}|\Lambda_i A|^{s-2}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A)\right|^2 \\
&\leqslant c\int_{B_r\setminus B_t}|\lambda Du|^2+|A|^{s-2}|\lambda Du|^2+|\lambda Du|^s \\
&\quad+c\int_{B_r\setminus B_t}\sum_{i=2}^{k}|\Lambda_i A|^{s-2}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^2 \\
&\quad+c\int_{B_r\setminus B_t}\sum_{i=2}^{k}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^s+c\int_{B_r\setminus B_t}\frac{|\lambda(u-a)|^2}{(r-t)^2}+\frac{|\lambda(u-a)|^s}{(r-t)^s} \\
&\quad+c\int_{B_r\setminus B_t}|A|^{s-2}\frac{|\lambda(u-a)|^2}{(r-t)^2}+c\sum_{i=2}^{k}\frac{M^{2(i-1)}|\Lambda_i A|^{s-2}}{(r-t)^2}\int_{B_r\setminus B_t}\left|\lambda(u-a)\right|^2 \\
&\quad+c\sum_{i=2}^{k}\frac{M^{s(i-1)}}{(r-t)^s}\int_{B_r\setminus B_t}\left|\lambda(u-a)\right|^s \\
&\quad+\frac{c(\Gamma,s,\gamma,n,N)}{2}\varepsilon\int_{B_r}\sum_{i=2}^{k}\eta^2|\Lambda_i A|^{s-2}\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^2 \\
&\quad+\frac{c(\Gamma,s,\gamma,n,N)}{2}\varepsilon\int_{B_r}\sum_{i=2}^{k}\eta^2\left|\Lambda_i(A+\lambda Du)-\Lambda_i A\right|^s
\end{aligned}
$$

$$+\frac{c(\Gamma,s,k,\gamma,n,N)\sum_{i=2}^{k}(M^{i(s-2)}M^{2(i-1)}+M^{s(i-1)})}{2\varepsilon}\int\limits_{B_r}\eta^2\big(|\lambda Du|^2+|\lambda Du|^s\big). \tag{3.10}$$

Choose $\varepsilon$ such that

$$c(\Gamma,s,\gamma,n,N)\varepsilon=1,$$

then choose $M$ such that

$$\frac{c(\Gamma,s,k,\gamma,n,N)\sum_{i=2}^{k}(M^{i(s-2)}M^{2(i-1)}+M^{s(i-1)})}{2\varepsilon}\leqslant\frac{1}{2}.$$

Then (3.10) implies

$$\begin{aligned}
&\int\limits_{B_t}\big[|\lambda Du|^2+|A|^{s-2}|\lambda Du|^2+|\lambda Du|^s\big]+\int\limits_{B_t}\sum_{i=2}^{k}\big|\Lambda_i(A+\lambda Du)-\Lambda_iA\big|^s\\
&\quad+\int\limits_{B_t}\sum_{i=2}^{k}|\Lambda_iA|^{s-2}\big|\Lambda_i(A+\lambda Du)-\Lambda_iA\big|^2\\
&\leqslant c\int\limits_{B_r\setminus B_t}\big[|\lambda Du|^2+|A|^{s-2}|\lambda Du|^2+|\lambda Du|^s\big]\\
&\quad+c\int\limits_{B_r\setminus B_t}\sum_{i=2}^{k}|\Lambda_iA|^{s-2}\big|\Lambda_i(A+\lambda Du)-\Lambda_iA\big|^2\\
&\quad+c\int\limits_{B_r\setminus B_t}\sum_{i=2}^{k}\big|\Lambda_i(A+\lambda Du)-\Lambda_iA\big|^s\\
&\quad+c\int\limits_{B_r\setminus B_t}\bigg[\frac{|\lambda(u-a)|^2}{(r-t)^2}+|A|^{s-2}\frac{|\lambda(u-a)|^2}{(r-t)^2}+\frac{|\lambda(u-a)|^s}{(r-t)^s}\bigg].
\end{aligned} \tag{3.11}$$

Here $c=c(M,s,\gamma,k,\Gamma,n,N)$. Add

$$\begin{aligned}
&c\bigg[\int\limits_{B_t}\big[|\lambda Du|^2+|A|^{s-2}|\lambda Du|^2+|\lambda Du|^s\big]+\int\limits_{B_t}\sum_{i=2}^{k}\big|\Lambda_i(A+\lambda Du)-\Lambda_iA\big|^s\\
&\quad+\int\limits_{B_t}\sum_{i=2}^{k}|\Lambda_iA|^{s-2}\big|\Lambda_i(A+\lambda Du)-\Lambda_iA\big|^2\bigg]
\end{aligned}$$

on both sides of (3.11), then divide $\lambda^2$ on both sides, the conclusion now follows from straightforward extension of Lemma 5.2 of [2] (see also [6, Chapter 5, Lemma 3.1]). □

## 4. Partial regularity

In this section, we prove partial regularity theorem. Consider the following quantity:

$$
\begin{aligned}
U(x,r) := {} & \fint_{B(x,r)} \left|Du-(Du)_{x,r}\right|^2 + \left|Du-(Du)_{x,r}\right|^s \\
& + \fint_{B(x,r)} \left|(Du)_{x,r}\right|^{s-2}\left|Du-(Du)_{x,r}\right|^2 \\
& + \fint_{B(x,r)} \left[\sum_{i=2}^{k}\left(\left|\Lambda_i(Du)_{x,r}\right|^{s-2}\left|\Lambda_i(Du)-\Lambda_i(Du)_{x,r}\right|^2\right)\right. \\
& \left. + \left|\Lambda_i(Du)-\Lambda_i(Du)_{x,r}\right|^s\right].
\end{aligned} \tag{4.1}
$$

We first prove the following decay estimate.

**Lemma 2.** *There exists $L_0 = L_0(n, N, s, k, \gamma, \Gamma) > 0$ such that for any weak solution $u$ of* (2.6) *with $\|Du\|_{L^\infty} \leqslant L_0$, and each $0 < \tau < \frac{1}{2}$ there exists $\varepsilon_0 = \varepsilon_0(\tau, L_0)$ such that for every $B(x,r) \subset \Omega$,*

$$U(x,r) \leqslant \varepsilon_0$$

*implies*

$$U(x,\tau r) \leqslant c_1\tau^2 U(x,r)$$

*for some $c_1 = c_1(L_0)$.*

**Proof.** We choose $c_1(L_0)$ later. Assume the lemma fails for some $L_0$ and $\tau$ fixed. By assumption there exists ball $B(x_m, r_m) \subset \Omega$ such that $\|Du\|_{L^\infty} \leqslant L_0$, $U(x_m, r_m) \to 0$ as $m \to \infty$, and

$$U(x_m, \tau r_m) > c_1\tau^2 U(x_m, r_m). \tag{4.2}$$

We will prove a contradiction to (4.2) for some sufficiently large $c_1 = c_1(L_0)$. In the following, we use $c$ denote a constant which may depend on $L_0$, but not on $\tau$ and $m$, and which may change from line to line. Let

$$\lambda_m^2 = U(x_m, r_m).$$

Then $\lambda_m \neq 0$ (otherwise $Du$ is constant on $B(x_m, r_m)$ and so both sides of (4.2) equal zero, contradicting (4.2)). Define

$$a_m = (u)_{x_m,r_m}, \qquad A_m = (Du)_{x_m,r_m}$$

and

$$v_m(y) = \frac{u(x_m + r_m y) - a_m - r_m A_m y}{\lambda_m r_m}$$

for all $y \in B_1(0)$. Then

$$
\begin{aligned}
&Dv_m(y) = \lambda_m^{-1}\big(Du(x_m + r_m y) - A_m\big),\\
&(Dv_m)_\tau = \lambda_m^{-1}\big((Du)_{x_m,\tau r_m} - A_m\big),\\
&Dv_m(y) - (Dv_m)_\tau = \lambda_m^{-1}\big(Du(x_m + r_m y) - (Du)_{x_m,\tau r_m}\big),\\
&(v_m)_1 = 0, \qquad (Dv_m)_1 = 0,\\
&1 = \fint_{B_1} |Dv_m|^2 + \lambda_m^{s-2}|Dv_m|^s + |A_m|^{s-2}|Dv_m|^2\\
&\quad + \fint_{B_1} \sum_{-i=2}^{k} |\Lambda_i A_m|^{s-2}\lambda_m^{-2}\big|\Lambda_i(\lambda_m Dv_m + A_m) - \Lambda_i A_m\big|^2\\
&\quad + \fint_{B_1} \sum_{i=2}^{k} \lambda_m^{-2}\big|\Lambda_i(\lambda_m Dv_m + A_m) - \Lambda_i A_m\big|^s
\end{aligned}
\tag{4.3}
$$

and $v_m$ satisfies for any $\varphi \in C_0^\infty(B_1)$

$$
\begin{aligned}
0 = &\int_{B_1} \big[DF^1(A_m + \lambda_m Dv_m) - DF^1(A_m)\big]D\varphi\\
&+ \int_{B_1} \sum_{i=2}^{k} \big\{\big[DF^i\big(\Lambda_i(A_m + \lambda_m Dv_m)\big) - DF^i(\Lambda_i A_m)\big]\\
&\cdot \big[\Lambda_{i-1}(A_m + \lambda_m Dv_m) \odot D\varphi\big]\big\}.
\end{aligned}
\tag{4.4}
$$

Passing to a subsequence we claim there exists a $n \times N$ matrix $A$ and some $v \in W^{1,2}(B_1)$ that

(i) $A_m \to A$ pointwise,
(ii) $v_m \rightharpoonup v$ in $W^{1,2}$,
(iii) $\lambda_m^{1-\frac{2}{s}} v_m \rightharpoonup 0$ in $W^{1,s}$ (if $s > 2$),
(iv) $|\Lambda_i A_m|^{\frac{s}{2}-1}\lambda_m^{i-1}\Lambda_i Dv_m \rightharpoonup 0$ in $L^2$, $2 \leqslant i \leqslant k$,
(v) $\lambda_m^{i-\frac{2}{s}}\Lambda_i Dv_m \rightharpoonup 0$ in $L^s$, $2 \leqslant i \leqslant k$,
(vi) $\lambda_m^\delta Dv_m \to 0$ a.e. for any $\delta > 0$.

The proof of (i)–(iii) and (vi) follows from [4, Section 5B]. To prove (iv) and (v), we observe the last equation in (4.3) implies $|\Lambda_i A_m|^{\frac{s}{2}-1}\lambda_m^{i-1}\Lambda_i Dv_m$ is uniformly bounded in $L^2$ and $\lambda_m^{i-\frac{2}{s}}\Lambda_i Dv_m$ is uniformly bounded in $L^s$ for $i = 1, \dots, k$. In fact, when $i = 1$, $Dv_m$ is uniformly bounded in $L^2$ by (4.3) and for $2 \leqslant i \leqslant k$,

$$
\begin{aligned}
\int_{B_1} \lambda_m^{2i-2}|\Lambda_i Dv_m|^2 \leqslant{}& 2\lambda_m^{-2}\int_{B_1} \big|\Lambda_i(A_m + \lambda_m Dv_m) - \Lambda_i A_m\big|^2\\
&+ 2\lambda_m^{-2}\int_{B_1} \left|\sum_{j=1}^{i-1} \Lambda_{i-j}A_m \odot \Lambda_j \lambda_m Dv_m\right|^2
\end{aligned}
$$

$$\leqslant 2\lambda_m^{-2} \int\limits_{B_1} \left|\Lambda_i(A_m + \lambda_m Dv_m) - \Lambda_i A_m\right|^2$$

$$+ c(k) \int\limits_{B_1} \sum_{j=1}^{i-1} |\Lambda_{i-j} A_m|^2 \lambda_m^{2j-2} |\Lambda_j Dv_m|^2. \tag{4.5}$$

By assumption, $|A_m| \leqslant L_0$, therefore $|\Lambda_i A_m| \leqslant c(L_0)$. This, together with (4.3), (4.5) and induction on $i$, implies $|\Lambda_i A_m|^{\frac{s}{2}-1} \lambda_m^{i-1} \Lambda_i Dv_m$ is uniformly bounded in $L^2$ for $i = 1, \dots, k$. The boundedness of $\lambda_m^{i-\frac{2}{s}} \Lambda_i Dv_m$ in $L^s$ can be proved similarly using induction on $i$. Once we have uniform boundedness of $|\Lambda_i A_m|^{\frac{s}{2}-1} \lambda_m^{i-1} \Lambda_i Dv_m$ in $L^2$ and uniform boundedness of $\lambda_m^{i-\frac{2}{s}} \Lambda_i Dv_m$ in $L^s$, one can apply the induction proof of [4, Section 5B] to show (iv) and (v) holds. Using (i)–(vi), it follows from the proof in [4, Section 5B] that the limit function $v$ satisfies the linear equation

$$0 = \int\limits_{B_1} \left[ D^2 F^1(A) Dv D\varphi + \sum_{i=2}^{k} D^2 F^i(\Lambda_i A)(\Lambda_{i-1} A \odot Dv) \Lambda_{i-1} A \odot D\varphi \right]. \tag{4.6}$$

By standard regularity results (see for example [6]) we have for any $\tau \in (0, 1)$ that

$$-\!\!\!\!\!\!\int\limits_{B_\tau} \left|Dv - (Dv)_\tau\right|^2 \leqslant c\tau^2 -\!\!\!\!\!\!\int\limits_{B_1} \left|Dv - (Dv)_1\right|^2 \leqslant c\tau^2 \tag{4.7}$$

and

$$\left|(Dv)_{2\tau} - (Dv)_\tau\right|^2 \leqslant c\tau^2, \tag{4.8}$$

where $c$ depends only on elliptic constant of system (4.6) and hence on $L_0$.

On rescaling (4.1) using (4.3) we have for any $\tau \in (0, \frac{1}{2})$ that

$$\begin{aligned} U(x_m, \tau r_m) = {} & -\!\!\!\!\!\!\int\limits_{B_\tau} \lambda_m^2 \left|Dv_m - (Dv_m)_\tau\right|^2 + \lambda_m^s \left|Dv_m - (Dv_m)_\tau\right|^s \\ & + -\!\!\!\!\!\!\int\limits_{B_\tau} \lambda_m^2 \left|A_m + \lambda_m (Dv_m)_\tau\right|^{s-2} \left|Dv_m - (Dv_m)_\tau\right|^2 \\ & + -\!\!\!\!\!\!\int\limits_{B_\tau} \sum_{i=2}^{k} \Big( \left|\Lambda_i\big(A_m + \lambda_m (Dv_m)_\tau\big)\right|^{s-2} \\ & \cdot \left|\Lambda_i(A_m + \lambda_m Dv_m) - \Lambda_i\big(A_m + \lambda_m (Dv_m)_\tau\big)\right|^2 \Big) \\ & + -\!\!\!\!\!\!\int\limits_{B_\tau} \sum_{i=2}^{k} \left|\Lambda_i(A_m + \lambda_m Dv_m) - \Lambda_i\big(A_m + \lambda_m (Dv_m)_\tau\big)\right|^s. \end{aligned}$$

Setting

$$w_m(z) = v_m(z) - (Dv_m)_\tau z - (v_m)_{2\tau}. \tag{4.9}$$

Then $w_m \in W^{1,\infty}$ is a weak solution to

$$
\begin{aligned}
0 = &\int_{B_1} DF^1\big(A_m + \lambda_m (Dv_m)_\tau + \lambda_m Dw_m\big) D\varphi \\
&+ \sum_{i=2}^{k} \int_{B_1} DF^i\big(\Lambda_i\big(A_m + \lambda_m (Dv_m)_\tau + \lambda_m Dw_m\big)\big) \\
&\cdot \Lambda_{i-1}\big(A_m + \lambda_m (Dv_m)_\tau + \lambda_m Dw_m\big) \odot D\varphi.
\end{aligned}
$$

Moreover, $\lambda_m \|Dw_m\|_{L^\infty} \leqslant 2\|Du\|_{L^\infty} = 2L_0$. Let $B_m = A_m + \lambda_m (Dv_m)_\tau$, then $|B_m| \leqslant L_0$. Apply Lemma 1 in Section 3 to $w_m$, we find a $L_0 = L_0(n, N, s, k, \gamma, \Gamma) > 0$ such that if $\lambda_m \|Dw_m\|_{L^\infty} \leqslant 2L_0$ and $B_m = A_m + \lambda_m (Dv_m)_\tau \leqslant L_0$, we have

$$
\begin{aligned}
\lambda_m^{-2} U(x_m, \tau r_m) = &\fint_{B_\tau} \big[|Dw_m|^2 + \lambda_m^{s-2} |Dw_m|^s + |B_m|^{s-2} |Dw_m|^2\big] \\
&+ \lambda_m^{-2} \sum_{i=2}^{k} \fint_{B_\tau} \big[|\Lambda_i B_m|^{s-2} \big|\Lambda_i (B_m + \lambda_m Dw_m) - \Lambda_i B_m\big|^2 \\
&+ \big|\Lambda_i (B_m + \lambda_m Dw_m) - \Lambda_i B_m\big|^s\big] \\
\leqslant &\frac{c}{\tau^2} \fint_{B_{2\tau}} |w_m|^2 + \frac{c}{\tau^s} \fint_{B_{2\tau}} \lambda_m^{s-2} |w_m|^s + \frac{c}{\tau^2} |B_m|^{s-2} \fint_{B_{2\tau}} |w_m|^2
\end{aligned}
$$

for some constant $c = c(L_0, n, N, s, k, \gamma, \Gamma)$. Passing to the limit, using (i)–(vi), the Poincaré inequality, (4.7) and (4.8), we obtain

$$
\begin{aligned}
\limsup_{m \to \infty} \frac{U(x_m, \tau r_m)}{\lambda_m^2} &\leqslant \frac{c}{\tau^2} \fint_{B_{2\tau}} \big|v - (v)_{2\tau} - (Dv)_\tau y\big|^2 \\
&\leqslant \frac{c}{\tau^2} \fint_{B_{2\tau}} \big|v - (v)_{2\tau} - (Dv)_{2\tau} y\big|^2 + \big|(Dv)_{2\tau} - (Dv)_\tau\big|^2 \\
&\leqslant c \fint_{B_{2\tau}} \big|Dv - (Dv)_{2\tau}\big|^2 + c\tau^2 \\
&\leqslant c_2(L_0) \tau^2.
\end{aligned}
$$

This contradicts (4.2) if $c_1(L_0)$ is chosen larger than $c_2(L_0)$. □

**Remark 2.** Using the fact that

$$
\begin{aligned}
&\|A_m + \lambda_m Dv_m\|_\infty \leqslant L_0, \qquad |A_m|_\infty \leqslant L_0, \\
&\lambda_m Dv_m \to 0 \quad \text{strongly in } L^2,
\end{aligned}
$$

we can immediately derive (4.6). Here we felt it might be of interest to point out further properties (i)–(vi).

A standard iteration and bootstrapping argument implies Theorem 1 with singular set $\Omega \sim \Omega_0$ where $\Omega_0$ is

$$\Omega_0 = \Big\{ x \in \Omega\colon \lim_{r\to 0} \fint_{B(x,r)} \big|Du - (Du)_{x,r}\big|^2 = 0 \text{ and}$$
$$\lim_{r\to 0} \fint_{B(x,r)} \big|\Lambda_i Du - \Lambda_i (Du)_{x,r}\big|^s = 0 \text{ for } i = 1, \dots, k \Big\}$$
$$= \Big\{ x \in \Omega\colon \lim_{r\to 0} \fint_{B(x,r)} \big|Du - (Du)_{x,r}\big|^2 = 0 \Big\}.$$

## 5. Reduced Hausdorff dimension for singular set

In this section, we follow Mingione and Kristensen's idea [9] to reduce the Hausdorff dimension of singular set for Lipschitz solutions. In [9], Mingione and Kristensen prove the singular set for Lipschitzian minimizers of strongly quasiconvex functionals has Hausdorff dimension strictly less than $n$. Their main idea is to show the singular set of a Lipschitzian minimizer is uniform porous. It then follows that the singular set has Hausdorff dimension strictly less than $n$. To prove uniform porosity of the singular set, their estimates rely crucially on the Caccioppoli inequalities. The main idea is that the Caccioppoli inequality implies certain Carleson type estimates, from which uniform porosity follows.

First we recall the definition of $(\lambda, \kappa)$-porous subset of $\mathbb{R}^n$ from [15]. We refer the reader to [11] for further information on porous subsets. Define

$$p(A, x, r) := \sup\big\{\rho\colon\ B(y, \rho) \subset B(x, r) \setminus A \text{ for some } y \in \mathbb{R}^n\big\}.$$

Then $0 \leqslant p(A, x, r) \leqslant r$ for all $x \in \mathbb{R}^n$ and $0 \leqslant p(A, x, r) \leqslant \frac{r}{2}$ for all $x \in A$.

**Definition 1.** For numbers $\kappa > 0$, $\lambda \in [0, \frac{1}{2}]$ we say that the subset $A \subset \mathbb{R}^n$ is $(\lambda, \kappa)$-porous provided $p(A, x, r) \geqslant \lambda r$ holds for all $x \in \mathbb{R}^n$ and all $r \in (0, \kappa)$.

Proof of the following lemma can be found in [15].

**Lemma 3.** *There exists a computable, strictly decreasing and continuous function $d_n : \mathbb{R}^+ \to \mathbb{R}^+$ with $d_n(0) = n$, such that whenever $A$ is a nonempty bounded $(\lambda, \kappa)$-porous subset of $\mathbb{R}^n$, where $\kappa > 0$ and $\lambda \in (0, \frac{1}{2})$, then $\dim_{\mathcal{H}} A \leqslant d_n(\lambda) < n$.*

**Proposition 1.** *Let $u \in W^{1,\infty}$ satisfies*

$$\int_{\Omega} \Bigg[ DF^1(Du)D\varphi + \sum_{i=2}^{k} DF^i(\Lambda_i Du)\Lambda_{i-1} Du \odot D\varphi \Bigg] dx = 0$$

*for any $\varphi \in C_0^\infty(\Omega)$. Then there exists $M_0 = M_0(n, N, s, k, \gamma, \Gamma) > 0$, such that if $\|Du\|_{L^\infty} \leqslant M_0$, then for every linear map $b(x) = u_0 + \langle z_0, x - x_0\rangle$ with $u_0 \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{n\times N}$ and $|u_0| + |z_0| \leqslant M_0$, and every ball $B(x_0, 2r) \subset \Omega$, we have*

$$\int_{B(x_0,r)} |Du - Db|^2 + |Du - Db|^s + |A|^{s-2} \int_{B(x_0,r)} |Du - Db|^2$$

$$+\sum_{i=2}^{k}|\Lambda_i A|^{s-2}\int\limits_{B(x_0,r)}\left|\Lambda_i(Du)-\Lambda_i(Db)\right|^2+\sum_{i=2}^{k}\int\limits_{B(x_0,r)}q\left|\Lambda_i(Du)-\Lambda_i(Db)\right|^s$$

$$\leqslant\frac{c}{r^2}\int\limits_{B(x_0,2r)}|u-b|^2+\frac{c}{r^s}\int\limits_{B(x_0,2r)}|u-b|^s$$

*where the constant $c=c(\|Du\|_{L^\infty},M_0,s,k,\gamma,\Gamma,n,N)$.*

**Proof.** Let $v(x)=u(x)-b(x)$. Then $v\in W^{1,\infty}$ satisfies

$$0=\int\limits_{\Omega}DF^1(Dv+Db)D\varphi+\sum_{i=2}^{k}DF^i\big(\Lambda_i(Dv+Db)\big)\Lambda_{i-1}(Dv+Db)\odot D\varphi$$

for any $\varphi\in C_0^\infty(\Omega)$ and $\|Dv\|_{L^\infty}\leqslant 2M_0$. We apply Lemma 1 to $v$ with $\lambda=1$, $a=0$, $A=Db=z_0$. The conclusion follows easily. □

If $u$ is a Lipschitzian global minimizer of (2.5), we have

**Proposition 2.** *Let $u\in W^{1,\infty}$ be a global minimizer of* (2.5). *For each $L>0$, there exists a constant $c=c(\|Du\|_{L^\infty},L,s,k,\gamma,\Gamma,n,N)$ such that for every linear map $b(x)=u_0+\langle z_0,x-x_0\rangle$ with $u_0\in\mathbb{R}^N$, $z_0\in\mathbb{R}^{n\times N}$ and $|u_0|+|z_0|\leqslant L$, and every ball $B(x_0,2r)\subset\Omega$, we have*

$$\int\limits_{B(x_0,r)}|Du-Db|^2+|Du-Db|^s+|Db|^{s-2}\int\limits_{B(x_0,r)}|Du-Db|^2$$

$$+\sum_{i=2}^{k}|\Lambda_i Db|^{s-2}\int\limits_{B(x_0,r)}\left|\Lambda_i(Du)-\Lambda_i(Db)\right|^2+\sum_{i=2}^{k}\int\limits_{B(x_0,r)}\left|\Lambda_i(Du)-\Lambda_i(Db)\right|^s$$

$$\leqslant\sum_{j=2}^{k}\frac{c}{r^j}\int\limits_{B(x_0,2r)}|u-b|^j+\sum_{j=1}^{k}\frac{c}{r^{2j}}\int\limits_{B(x_0,2r)}|u-b|^{2j}+\sum_{j=1}^{k}\frac{c}{r^{sj}}\int\limits_{B(x_0,2r)}|u-b|^{sj}.$$

**Proof.** The main idea follows Evans [2, Lemma 3.1], we sketch some details for reader's convenience. We may assume $x_0=0$. Let $r\leqslant t<\tau\leqslant 2r$ and choose $\varsigma\in C_0^\infty(\Omega;\mathbb{R})$ satisfying

$$\varsigma\equiv 1\quad\text{on } B_t,\qquad \varsigma\equiv 0\quad\text{on }\Omega\setminus B_\tau,$$
$$0\leqslant\varsigma\leqslant 1,\quad |D\varsigma|\leqslant\frac{C}{\tau-t}.$$

Define

$$\phi\equiv\varsigma(u-b),\qquad \psi\equiv(1-\varsigma)(u-b);$$

then

$$D\phi+D\psi=Du-Db.$$

Since $\operatorname{supp}\varsigma\subset B_\tau$, hypotheses (H3), (H4) and (2.3) imply

$$
\begin{aligned}
\int_{B_\tau} F(Db+D\phi) &= \int_{B_\tau} F_1(Db+D\phi) + \sum_{i=2}^{k} F_i\big(\Lambda_i(Db+D\phi)\big) \\
&= \int_{B_\tau} F_1(Db) + \sum_{i=2}^{k} F_i(\Lambda_i Db) \\
&\quad + \int_{B_\tau} \int_0^1 (1-\tau) D^2 F_1(Db+\tau D\phi) D\phi D\phi \, d\tau \\
&\quad + \sum_{i=2}^{k} \int_{B_\tau} \int_0^1 (1-\tau) D^2 F_i\big(\Lambda_i Db + \tau\big(\Lambda_i(Db+D\phi) - \Lambda_i Db\big)\big) \\
&\quad \cdot \big(\Lambda_i(Db+D\phi) - \Lambda_i Db\big)\big(\Lambda_i(Db+D\phi) - \Lambda_i Db\big) \\
&\geqslant \int_{B_\tau} F_1(Db) + \sum_{i=2}^{k} F_i(\Lambda_i Db) \\
&\quad + \gamma \int_{B_\tau} \int_0^1 (1-\tau)\big(1 + |Db+\tau D\phi|^{s-2}\big)|D\phi|^2 \, d\tau \\
&\quad + \gamma \sum_{i=2}^{k} \int_{B_\tau} \int_0^1 \big[(1-\tau)\big(\big|\Lambda_i Db + \tau\big(\Lambda_i(Db+D\phi) - \Lambda_i Db\big)\big|^{s-2}\big)\, d\tau \\
&\quad \cdot \big|\Lambda_i(Db+D\phi) - \Lambda_i Db\big|^2\big] \\
&\geqslant \int_{B_\tau} F(Db) + \sigma\gamma \int_{B_\tau} |D\phi|^2 + |D\phi|^s + |Db|^{s-2}|D\phi|^2 \\
&\quad + \sigma\gamma \sum_{i=2}^{k} \int_{B_\tau} |\Lambda_i Db|^{s-2}\big|\Lambda_i(Db+D\phi) - \Lambda_i Db\big|^2 \\
&\quad + \big|\Lambda_i(Db+D\phi) - \Lambda_i Db\big|^s. \qquad (5.1)
\end{aligned}
$$

In the last inequality, we use Lemma 8.1 of [2]: there exists $\sigma > 0$ such that, for any two $n \times N$ matrices $A$ and $B$,

$$
\sigma\big(|A|^{s-2} + |B|^{s-2}\big) \leqslant \int_0^1 (1-\tau)|A+\tau B|^{s-2} \, d\tau.
$$

On the other hand,

$$
\int_{B_\tau} F(Db+D\phi) = \int_{B_\tau} F(Du - D\psi)
$$

$$
\begin{aligned}
&= \int_{B_\tau} F_1(Du - D\psi) + \int_{B_\tau} \sum_{i=2}^{k} F_i\big(\Lambda_i(Du - D\psi)\big) \\
&= \int_{B_\tau} F_1(Du) - DF_1(Du)D\psi + \int_{B_\tau} D^2 F_1\big(Du - \tau(x)D\psi\big)D\psi D\psi \\
&\quad + \int_{B_\tau} \sum_{i=2}^{k} F_i(\Lambda_i Du) - DF_i(\Lambda_i Du)\big(\Lambda_i Du - \Lambda_i(Du - D\psi)\big) \\
&\quad + \int_{B_\tau} \sum_{i=2}^{k} D^2 F_i\big(\Lambda_i Du + \tau(x)\big(\Lambda_i Du - \Lambda_i(Du - D\psi)\big)\big) \\
&\quad \cdot \big(\Lambda_i Du - \Lambda_i(Du - D\psi)\big)\big(\Lambda_i Du - \Lambda_i(Du - D\psi)\big) \\
&\leqslant \int_{B_\tau} F_1(Du) + \sum_{i=2}^{k} F_i(\Lambda_i Du) \\
&\quad - \int_{B_\tau} DF_1(Du)D\psi - \int_{B_\tau} \sum_{i=2}^{k} DF_i(\Lambda_i Du)\big(\Lambda_i Du - \Lambda_i(Du - D\psi)\big) \\
&\quad + c(\Gamma, s) \int_{B_\tau} \big(1 + |Du|^{s-2} + |D\psi|^{s-2}\big)|D\psi|^2 \\
&\quad + c(\Gamma, s) \sum_{i=2}^{k} \int_{B_\tau} \big(|\Lambda_i Du|^{s-2} + \big|\Lambda_i Du - \Lambda_i(Du - D\psi)\big|^{s-2}\big) \\
&\quad \cdot \big|\Lambda_i Du - \Lambda_i(Du - D\psi)\big|^2. \qquad (5.2)
\end{aligned}
$$

Since $u$ is a minimizer, $\operatorname{supp}\phi \subset B_\tau$, we have

$$
\begin{aligned}
\int_{B_\tau} F(Du) &\leqslant \int_{B_\tau} F(Du - D\phi) \\
&= \int_{B_\tau} F(Db + D\psi) \\
&= \int_{B_\tau} F_1(Db + D\psi) + \int_{B_\tau} \sum_{i=2}^{k} F_i\big(\Lambda_i(Db + D\psi)\big) \\
&= \int_{B_\tau} F_1(Db) + DF_1(Db)D\psi + \int_{B_\tau} \sum_{i=2}^{k} F_i(\Lambda_i Db) \\
&\quad + \int_{B_\tau} D^2 F_1\big(Db + \tau(x)D\psi\big)D\psi D\psi
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{B_\tau}\sum_{i=2}^{k} DF_i(\Lambda_i Db)\big(\Lambda_i(Db+D\psi)-\Lambda_i Db\big)\\
&+\int_{B_\tau}\sum_{i=2}^{k} D^2F_i\big(\Lambda_i Db+\tau(x)\big(\Lambda_i(Db+D\psi)-\Lambda_i Db\big)\big)\\
&\cdot\big(\Lambda_i(Db+D\psi)-\Lambda_i Db\big)\big(\Lambda_i(Db+D\psi)-\Lambda_i Db\big)\\
\leqslant{}&\int_{B_\tau} F(Db)+\int_{B_\tau} DF_1(Db)D\psi\\
&+\int_{B_\tau}\sum_{i=2}^{k} DF_i(\Lambda_i Db)\big(\Lambda_i(Db+D\psi)-\Lambda_i Db\big)\\
&+c(\Gamma,s)\int_{B_\tau}\big(1+|Db|^{s-2}+|D\psi|^{s-2}\big)|D\psi|^2\\
&+c(\Gamma,s)\sum_{i=2}^{k}\int_{B_\tau}\big(|\Lambda_i Db|^{s-2}+\big|\Lambda_i Db-\Lambda_i(Db+D\psi)\big|^{s-2}\big)\\
&\cdot\big|\Lambda_i Db-\Lambda_i(Db+D\psi)\big|^2.
\end{aligned}\tag{5.3}
$$

(5.2), (5.3) combined with (5.1) give

$$
\begin{aligned}
\sigma\gamma&\int_{B_\tau}\big(|D\phi|^2+|D\phi|^s+|Db|^{s-2}|D\phi|^2\big)\\
&+\sigma\gamma\sum_{i=2}^{k}\int_{B_\tau}|\Lambda_i Db|^{s-2}\big|\Lambda_i(Db+D\phi)-\Lambda_i Db\big|^2\\
&+\sigma\gamma\sum_{i=2}^{k}\int_{B_\tau}\big|\Lambda_i(Db+D\phi)-\Lambda_i Db\big|^s\\
&\leqslant\int_{B_\tau} DF_1(Db)D\psi-\int_{B_\tau} DF_1(Du)D\psi\\
&-\int_{B_\tau}\sum_{i=2}^{k} DF_i(\Lambda_i Du)\big(\Lambda_i Du-\Lambda_i(Du-D\psi)\big)\\
&+\int_{B_\tau}\sum_{i=2}^{k} DF_i(\Lambda_i Db)\big(\Lambda_i(Db+D\psi)-\Lambda_i Db\big)\\
&+c(\Gamma,s)\int_{B_\tau}\big(1+|Db|^{s-2}+|D\psi|^{s-2}\big)|D\psi|^2
\end{aligned}
$$

$$
\begin{aligned}
&+c(\Gamma,s)\sum_{i=2}^{k}\int_{B_\tau}\bigl(|\Lambda_i Db|^{s-2}+\bigl|\Lambda_i Db-\Lambda_i(Db+D\psi)\bigr|^{s-2}\bigr)\\
&\cdot\bigl|\Lambda_i Db-\Lambda_i(Db+D\psi)\bigr|^2\\
&+c(\Gamma,s)\int_{B_\tau}\bigl(1+|Du|^{s-2}+|D\psi|^{s-2}\bigr)|D\psi|^2\\
&+c(\Gamma,s)\sum_{i=2}^{k}\int_{B_\tau}\bigl(|\Lambda_i Du|^{s-2}+\bigl|\Lambda_i Du-\Lambda_i(Du-D\psi)\bigr|^{s-2}\bigr)\\
&\cdot\bigl|\Lambda_i Du-\Lambda_i(Du-D\psi)\bigr|^2\\
=\;& R.
\end{aligned}
\tag{5.4}
$$

The right-hand side can be estimated by

$$
\begin{aligned}
R\leqslant&\int_{B_\tau}\int_0^1\bigl|D^2F_1\bigl(Du+\tau(Db-Du)\bigr)\bigr|\,d\tau\,|Db-Du||D\psi|\\
&+\int_{B_\tau}\sum_{i=2}^{k}\bigl|DF_i(\Lambda_i Du)\Lambda_{i-1}Du-DF_i(\Lambda_i Db)\Lambda_{i-1}Db\bigr||D\psi|\\
&+\int_{B_\tau}\sum_{i=2}^{k}\bigl|DF_i(\Lambda_i Du)\bigr|\left|\sum_{j=2}^{i}\Lambda_{i-j}Du\odot\Lambda_j(-D\psi)\right|\\
&+\int_{B_\tau}\sum_{i=2}^{k}\bigl|DF_i(\Lambda_i Db)\bigr|\left|\sum_{j=2}^{i}\Lambda_{i-j}Db\odot\Lambda_j(D\psi)\right|\\
&+c(\Gamma,s)\int_{B_\tau}\bigl(1+|Db|^{s-2}+|Du|^{s-2}+|D\psi|^{s-2}\bigr)|D\psi|^2\\
&+c(\Gamma,s)\sum_{i=2}^{k}\int_{B_\tau}\left(|\Lambda_i Db|^{s-2}+\left|\sum_{j=1}^{i}\Lambda_{i-j}Db\odot\Lambda_j(D\psi)\right|^{s-2}\right)\\
&\cdot\left|\sum_{j=1}^{i}\Lambda_{i-j}Db\odot\Lambda_j(D\psi)\right|^2\\
&+c(\Gamma,s)\sum_{i=2}^{k}\int_{B_\tau}|\Lambda_i Du|^{s-2}\left|\sum_{j=1}^{i}\Lambda_{i-j}Du\odot\Lambda_j(-D\psi)\right|^2\\
&+\left|\sum_{j=1}^{i}\Lambda_{i-j}Du\odot\Lambda_j(-D\psi)\right|^{s}.
\end{aligned}
\tag{5.5}
$$

Now since $|Db| \leqslant L$, we have

$$|Du|^{s-2} \leqslant C(L,s)\big(1+|Du-Db|^{s-2}\big)$$

and

$$\big|DF_i(\Lambda_i Du)\Lambda_{i-1}Du - DF_i(\Lambda_i Db)\Lambda_{i-1}Db\big| \leqslant c\big(L, \|\nabla u\|_{L^\infty}\big)|Du-Db|.$$

Furthermore $D\phi = Du - Db$ on $B_t$ and $\psi \equiv 0$ on $B_t$. These together with elementary inequalities

$$\alpha^{s-1}\beta \leqslant \alpha^s + \beta^s, \quad \alpha^{s-2}\beta^2 \leqslant \alpha^s + \beta^s \quad (\alpha,\beta \geqslant 0)$$

and (5.4), (5.5) imply

$$\begin{aligned}
&\sigma\gamma \int_{B_t} |Du-Db|^2 + |Du-Db|^s + |Db|^{s-2}|Du-Db|^2 \\
&\quad + \sigma\gamma \sum_{i=2}^k \int_{B_t} |\Lambda_i Db|^{s-2}\big|\Lambda_i(Du) - \Lambda_i Db\big|^2 + \sigma\gamma \sum_{i=2}^k \int_{B_t} \big|\Lambda_i(Du) - \Lambda_i Db\big|^s \\
&\leqslant c \int_{B_\tau \setminus B_t} \big(1+|Db-Du|^{s-2}\big)|Db-Du||D\psi| \\
&\quad + c \int_{B_\tau \setminus B_t} \big(1+|Du-Db|^{s-2} + |D\psi|^{s-2}\big)|D\psi|^2 \\
&\quad + c \int_{B_\tau \setminus B_t} \sum_{j=2}^k |\Lambda_j D\psi| + \sum_{j=1}^k |\Lambda_j D\psi|^2 + \sum_{j=1}^k |\Lambda_j D\psi|^s \\
&\leqslant c \int_{B_\tau \setminus B_t} \big(|Du-Db|^2 + |Du-Db|^s\big) + c \int_{B_\tau \setminus B_t} \sum_{j=2}^k |D\psi|^j + \sum_{j=1}^k \big(|D\psi|^{2j} + |D\psi|^{sj}\big)
\end{aligned} \tag{5.6}$$

for some constant $c = c(\Gamma, s, k, L, \|Du\|_{L^\infty})$. Since

$$|D\psi| \leqslant c|Du-Db| + \frac{c}{\tau - t}|u-b|,$$

from (5.6) it follows that

$$\begin{aligned}
&\sigma\gamma \int_{B_t} |Du-Db|^2 + |Du-Db|^s + |Db|^{s-2}|Du-Db|^2 \\
&\quad + \sigma\gamma \int_{B_t} |\Lambda_i Db|^{s-2}\big|\Lambda_i(Du) - \Lambda_i Db\big|^2 + \big|\Lambda_i(Du) - \Lambda_i Db\big|^s \\
&\leqslant c\big(\|Du\|_{L^\infty}, L, s, k, \Gamma\big) \int_{B_\tau \setminus B_t} \big(|Du-Db|^2 + |Du-Db|^s\big)\,dx \\
&\quad + c \int_{B_{2r}} \sum_{j=2}^k \frac{|u-b|^j}{(s-t)^j} + \sum_{j=1}^k \bigg(\bigg(\frac{|u-b|}{(s-t)}\bigg)^{2j} + \bigg(\frac{|u-b|}{(s-t)}\bigg)^{sj}\bigg).
\end{aligned}$$

Consequently

$$
\begin{aligned}
&\int\limits_{B_t} \big(|Du - Db|^2 + |Du - Db|^s + |Db|^{s-2}|Du - Db|^2\big) \\
&\qquad + \int\limits_{B_t} \sum_{j=2}^{k} \big(|\Lambda_i Db|^{s-2}\big|\Lambda_i(Du) - \Lambda_i Db\big|^2 + \big|\Lambda_i(Du) - \Lambda_i Db\big|^s\big) \\
&\quad \leqslant \theta \int\limits_{B_\tau} \big(|Du - Db|^2 + |Du - Db|^s + |Db|^{s-2}|Du - Db|^2\big) \\
&\qquad + \theta \int\limits_{B_\tau} \sum_{j=2}^{k} \big(|\Lambda_i Db|^{s-2}\big|\Lambda_i(Du) - \Lambda_i Db\big|^2 + \big|\Lambda_i(Du) - \Lambda_i Db\big|^s\big) \\
&\qquad + c \int\limits_{B_{2r}} \sum_{j=2}^{k} \frac{|u-b|^j}{(\tau - t)^j} + \sum_{j=1}^{k} \left( \left( \frac{|u-b|}{(\tau - t)} \right)^{2j} + \left( \frac{|u-b|}{(\tau - t)} \right)^{sj} \right).
\end{aligned}
$$

Here $\theta < 1$. The conclusion of the proposition then follows from the following extension of Lemma 5.2 in [2]. $\square$

**Lemma 4.** *Let $f : [\frac{r}{2}, r] \to [0, \infty)$ be bounded and satisfy*

$$
f(t) \leqslant \theta f(\tau) + \sum_{j=2}^{k} \frac{A_j}{(\tau - t)^j} + \sum_{j=1}^{k} \left( \frac{B_j}{(\tau - t)^{2j}} + \frac{C_j}{(\tau - t)^{sj}} \right) \tag{5.7}
$$

*for some $\theta < 1$, and all $\frac{r}{2} \leqslant t < \tau \leqslant r$, then there exists a constant $c = c(s, \theta, k)$ such that*

$$
f(r) \leqslant c \left[ \sum_{j=2}^{k} \frac{A_j}{r^j} + \sum_{j=1}^{k} \left( \frac{B_j}{r^{2j}} + \frac{C_j}{r^{sj}} \right) \right].
$$

**Proof.** This is a variant of Lemma 3.1 in Chapter 5 in [6] and Lemma 5.2 in [2]. Set

$$
t_l \equiv r\left(1 - \frac{\chi^l}{2}\right) \quad \text{for } l = 0, 1, 2, \ldots,
$$

where $0 < \chi < 1$ is a constant chosen later. Then

$$
t_0 = \frac{r}{2}, \qquad \lim_{l \to \infty} t_l = r, \qquad t_{l+1} - t_l = \frac{r}{2}(1 - \chi)\chi^l.
$$

According to (5.7) we have

$$
\begin{aligned}
f(t_l) &\leqslant \theta f(t_{l+1}) + \sum_{j=2}^{k} \frac{A_j}{(t_{l+1} - t_l)^j} + \sum_{j=1}^{k} \left( \frac{B_j}{(t_{l+1} - t_l)^{2j}} + \frac{C_j}{(t_{l+1} - t_l)^{sj}} \right) \\
&\leqslant \theta f(t_{l+1}) + \frac{c}{\chi^{lsk}} \left[ \sum_{j=2}^{k} \frac{A_j}{r^j} + \sum_{j=1}^{k} \left( \frac{B_j}{r^{2j}} + \frac{C_j}{r^{sj}} \right) \right]
\end{aligned}
$$

for $l = 0, 1, 2, \ldots$. By iteration, we get

$$f(t_0) \leqslant \theta^l f(t_l) + c\left[\sum_{j=2}^{k} \frac{A_j}{r^j} + \sum_{j=1}^{k}\left(\frac{B_j}{r^{2j}} + \frac{C_j}{r^{sj}}\right)\right]\sum_{i=0}^{l-1}\left(\frac{\theta}{\chi^{sk}}\right)^i.$$

Choose $\chi < 1$ such that $\frac{\theta}{\chi^{sk}} < 1$ then send $l$ to infinity. This completes the proof. □

We quote the following lemma from [9].

**Lemma 5.** *(See [9, Lemma 2.2].) There exists a constant $c = c(n)$ depending only on $n$ such that for any ball $B(x_0, 2R) \subset \mathbb{R}^n$ and any $w \in W^{1,2}(B(x_0, 2R), \mathbb{R}^N)$ the inequality*

$$\int\limits_{B(x_0,R)} \int\limits_0^R \fint\limits_{B(x,r)} \left|\frac{w(y) - (w)_{x,r} - (Dw)_{x,r}(y-x)}{r}\right|^2 dy\,\frac{dr}{r}\,dx \leqslant c \int\limits_{B(x_0,2R)} |Dw|^2\,dx \tag{5.8}$$

*holds.*

Proposition 1 and Lemma 5 give

**Proposition 3.** *Let $u \in W^{1,\infty}$ satisfies*

$$\int\limits_{\Omega}\left[DF^1(Du)D\varphi + \sum_{i=2}^{k} DF^i(\Lambda_i Du)\Lambda_{i-1}Du \odot D\varphi\right]dx = 0$$

*for any $\varphi \in C_0^\infty(\Omega)$. There exists a constant $M_0 = M_0(n, N, s, k, \gamma, \Gamma)$ such that for any weak solution $u$ with $\|Du\|_{L^\infty} \leqslant M_0$, there exist two constants*

$$C_0 = C_0\big(n, N, s, k, \gamma, \Gamma, \|Du\|_{L^\infty}\big), \qquad R_0 = R_0(C_0),$$

*such that for all balls $B(x_0, 4R) \subset \Omega$ with radii $R \leqslant R_0$, the inequality*

$$\fint\limits_{B(x_0,R)} \int\limits_0^R U(x,r)\frac{dr}{r}\,dx \leqslant C_0$$

*holds. Here $U(x,r)$ is defined by* (4.1).

**Proof.** For $B(x, 2r) \subset \Omega$, consider linear map $b(y) = (u)_{x,2r} + (Du)_{x,2r}(y-x)$. Then

$$\sup_{B(x,2r)} \left|u - (u)_{x,2r}\right| \leqslant 2mr, \quad m = \|Du\|_{L^\infty}.$$

By Proposition 1,

$$\begin{aligned}
U(x,r) &\leqslant \frac{c(m)}{r^2} \fint\limits_{B(x,2r)} |u-b|^2 + \frac{c(m)}{r^s} \fint\limits_{B(x,2r)} |u-b|^s \\
&\leqslant \frac{c(m)}{r^2}\big(1 + (2m)^{s-2}\big) - \fint\limits_{B(x,2r)} |u-b|^2 \\
&= \frac{c(m,s)}{r^2} - \fint\limits_{B(x,2r)} \left|u - (u)_{x,2r} - (Du)_{x,2r}(y-x)\right|^2.
\end{aligned}$$

Since

$$\begin{aligned}&\int_0^R \fint_{B(x,2r)} \left|u-(u)_{x,2r}-(Du)_{x,2r}(y-x)\right|^2 dy\,\frac{dr}{r^3}\\&\quad=4\int_0^{2R} \fint_{B(x,r)} \left|u-(u)_{x,r}-(Du)_{x,r}(y-x)\right|^2 dy\,\frac{dr}{r^3},\end{aligned}$$

we get by Lemma 5,

$$\begin{aligned}&\fint_{B(x_0,R)} \int_0^R U(x,r)\frac{dr}{r}\,dx\\&\quad\leqslant c(m)\fint_{B(x_0,R)} \int_0^R \fint_{B(x,2r)} \left|u-(u)_{x,2r}-(Du)_{x,2r}(y-x)\right|^2 dy\,\frac{dr}{r^3}\,dx\\&\quad\leqslant c(n,m)\fint_{B(x_0,2R)} \int_0^{2R} \fint_{B(x,r)} \left|u-(u)_{x,r}-(Du)_{x,r}(y-x)\right|^2 dy\,\frac{dr}{r^3}\,dx\\&\quad\leqslant c\fint_{B(x_0,4R)} \left|Du(x)\right|^2 dx\\&\quad\leqslant c. \qquad \square\end{aligned}$$

One can then follow the proof of [9, Section 5] line by line to show the porosity of the singular set of a Lipschitz weak solution.

**Proposition 4.** *Let $u \in W^{1,\infty}$ satisfies*

$$\int_\Omega \left[DF^1(Du)D\varphi + \sum_{i=2}^k DF^i(\Lambda_i Du)\Lambda_{i-1}Du \odot D\varphi\right] dx = 0$$

*for any $\varphi \in C_0^\infty(\Omega)$. There exists constant $M = M(n, N, s, k, \gamma, \Gamma) > 0$, such that for any $u$ satisfying $\|Du\|_{L^\infty} \leqslant M$, there exists constant*

$$\lambda = \lambda\big(n, N, s, k, \gamma, \Gamma, \|Du\|_{L^\infty}\big) \in \left(0, \frac{1}{2}\right)$$

*such that for each $\Omega' \Subset \Omega$ there is a $\kappa > 0$ for which $\Omega' \cap \Omega_0$ is $(\lambda, \kappa)$-porous.*

The conclusion of Theorem 1 follows directly from Proposition 4 and Lemma 3.
The case of global minimizer can be proved similarly.

**Remark 3.** Under the assumption that $F$ is strongly quasiconvex with the following growth assumption

$$\left|D^2F(P)\right| \leqslant c\big(|P|^{q-2}+1\big)$$

for some $q > 2$, then for any global Lipschitzian minimizer of

$$I(u) = \int_{\Omega} F(Du),$$

one can prove the following Caccioppoli inequality

$$\int_{B(x_0,r)} |Du - Db|^2 \leqslant \frac{c}{r^2} \int_{B(x_0,2r)} |u-b|^2 + \frac{c}{r^q} \int_{B(x_0,2r)} |u-b|^q$$

using the same approach of Proposition 2. It then follows for this general case, we also have reduced dimension estimate on singular set for Lipschitzian minimizers.

## Acknowledgments

S.C.'s research is partially supported by IRGP grant from Michigan State University. X.Y.'s research is partially supported by NSF grant DMS 0431710 and IRGP grant from Michigan State University. X.Y. thanks the Institute for Applied Math. at Bonn for hospitality where part of this work was written.

## References

[1] E. Acerbi, N. Fusco, Semicontinuity problems in the calculus of variations, Arch. Ration. Mech. Anal. 86 (1984) 125–145.
[2] L.C. Evans, Quasiconvexity and partial regularity in the calculus of variations, Arch. Ration. Mech. Anal. 95 (1986) 227–252.
[3] N. Fusco, J. Hutchinson, $C^{1,\alpha}$ partial regularity of functions minimising quasi-convex integrals, Manuscripta Math. 54 (1985) 121–143.
[4] N. Fusco, J. Hutchinson, Partial regularity in problems motivated by nonlinear elasticity, SIAM J. Math. Anal. 72 (6) (1991) 1516–1551.
[5] N. Fusco, J. Hutchinson, Partial regularity and everywhere continuity for a model problem from nonlinear elasticity, J. Aust. Math. Soc. Ser. A 57 (2) (1994) 158–169.
[6] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton Univ. Press, Princeton, NJ, 1983.
[7] M. Giaquinta, G. Modica, Partial regularity of minimizers of quasi-convex integrals, Ann. Inst. H. Poincaré 3 (1986) 185–208.
[8] J. Kristensen, G. Mingione, The singular set of minima of integral functionals, Arch. Ration. Mech. Anal. 180 (3) (2006) 331–398.
[9] J. Kristensen, G. Mingione, The singular set of Lipschitzian minima of multiple integrals, Arch. Ration. Mech. Anal., in press.
[10] J. Kristensen, A. Taheri, Partial regularity of strong local minimizers in the multi-dimensional calculus of variations, Arch. Ration. Mech. Anal. 170 (2003) 63–89.
[11] P. Mattila, Geometry of Sets and Measures in the Euclidean Spaces, Cambridge Univ. Press, 1995.
[12] C.B. Morrey, Multiple Integrals in the Calculus of Variations, Springer-Verlag, New York, 1966.
[13] R. Moser, Vanishing mean oscillation and regularity in the calculus of variations, MPI, preprint, 2001.
[14] S. Müller, V. Šverák, Convex integration for Lipschitz mappings and counterexamples to regularity, Ann. of Math. (2) 157 (2003) 715–742.
[15] A. Salli, On the Minkowski dimension of strongly porous fractal sets in $\mathbb{R}^n$, Proc. London Math. Soc. (3) 62 (1991) 353–372.
[16] Lászlo Székelyhidi Jr., The regularity of critical points of polyconvex functionals, Arch. Ration. Mech. Anal. 172 (1) (2004) 133–152.